

MODULARITY OF CALABI-YAU VARIETIES

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ABSTRACT. In this paper we discuss recent progress on the modularity of Calabi-Yau varieties. We focus mostly on the case of surfaces and threefolds. We will also discuss some progress on the structure of the L -function in connection with mirror symmetry. Finally, we address some questions and open problems.

1. INTRODUCTION

In this paper we discuss recent progress on the modularity of Calabi-Yau varieties. We first explain what we mean by modularity. A d -dimensional variety X/\mathbb{Q} is called modular if the L -function of the Galois-representation on $H_{\text{ét}}^d(X, \mathbb{Q})$ equals the product of L -functions of modular forms up to the factors associated to the bad primes. In practice, if d is even then $H_{\text{ét}}^d(X, \mathbb{Q})$ contains many classes of $d/2$ -dimensional cycles. The Galois representation on the subspace generated by these classes is very often easy to calculate. In this case we consider only the subrepresentation on the orthogonal complement of the images of the algebraic cycles.

Almost trivially, the modular curves associated to subgroups of $\text{SL}(2, \mathbb{Z})$ of finite index are modular. The first non-trivial example are the elliptic curves over \mathbb{Q} with complex multiplication. Deuring [Deu53] proved that their L -series is the L -series of a Größencharacter. One can associate a modular form to such an L -series by the work of Hecke.

It has been a long standing conjecture that all elliptic curve over \mathbb{Q} are modular. This conjecture was known as the Taniyama-Shimura-Weil conjecture. Wiles [Wil95] proved a large part of this conjecture. A few years later Breuil, Conrad, Diamond and Taylor [BCDT01] completed the proof. In this paper we will concentrate on Calabi-Yau varieties of dimension two and higher.

In Section 2 we discuss some preliminaries, such as the Weil conjectures, étale cohomology and the definition of an L -function of a Galois representation. In Section 3 we explain how one can associate a Galois representation to a modular form. Furthermore, we discuss several general conjectures concerning the modularity and automorphy of Galois representations, namely the Fontaine-Mazur conjectures and we very briefly describe the Langlands' programme.

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In Section 4 we come back to the elliptic curve case in more detail and discuss a similar result for rigid Calabi-Yau threefolds. In Section 5 we discuss some examples of modular varieties, where we mainly focus on singular $K3$ -surfaces and Calabi-Yau threefolds, although we also consider some higher dimensional examples.

In Section 6 we discuss some recent progress on determining the structure of the L -function of Calabi-Yau varieties and the L -function of the mirror partner. Here we sketch a new approach to compute the zeta function of the one-parameter quintic family and its mirror. Finally, in Section 7 we address some open questions.

2. VARIETIES OVER \mathbb{Q} .

We shall discuss first varieties which are defined over the field \mathbb{Q} of rational numbers. Later we shall also, more generally, consider varieties defined over number fields. Let X be a d -dimensional projective variety defined over \mathbb{Q} , i.e., a variety defined by the vanishing of a finite number of homogeneous polynomials with rational coefficients.

Let p be a prime number. We say that X has *good reduction* at p if there exists a variety X'/\mathbb{Q}_p such that X and X' are isomorphic over \mathbb{Q}_p and the reduction of X' modulo p is smooth. If this is the case then we take X_p to be the reduction of X' . We call such p a *good prime* (for X). If p does not satisfy the above mentioned property then we say that X has *bad reduction* at p . Such a p is called a *bad prime*. The number of bad primes is always finite.

If we fix a model of X over \mathbb{Z} then for all but finitely many primes p we have another way to define X_p , i.e., we have that X_p is isomorphic to $X \times_{\text{Spec } \mathbb{Z}} \mathbb{F}_p$, provided that the latter is smooth.

If E is an elliptic curve over \mathbb{Q} we can find a model over \mathbb{Z} such that for all good primes p the curve $E \times_{\text{Spec } \mathbb{Z}} \mathbb{F}_p$ is smooth. However, in general one cannot expect to find one model that suffices for all good primes.

If we consider a variety X defined over a field K that is not algebraically closed, then we indicate by \overline{X} the same variety considered over the algebraic closure \overline{K} .

2.1. The zeta function. Let $q = p^m$ be a prime power and let X/\mathbb{F}_q be a variety. Set

$$N_{q^r} = \#X(\mathbb{F}_{q^r})$$

be the number of points of X over \mathbb{F}_{q^r} . Then the zeta function of X/\mathbb{F}_q is defined by

$$Z_q(t) = \exp \left(\sum_{r=1}^{\infty} N_{q^r} \frac{t^r}{r} \right) \in \mathbb{Q}[[t]].$$

Weil [Weil49] has made a series of famous conjectures concerning this function:

Theorem 2.1 (Weil conjectures). *Let X be a smooth d -dimensional projective variety defined over the field \mathbb{F}_q , $q = p^r$. The function $Z_q(t)$ satisfies the following properties*

- (1) Rationality, i.e.

$$Z_q(t) = \frac{P_q(t)}{Q_q(t)} \text{ for polynomials } P_q(t), Q_q(t) \in \mathbb{Z}[t].$$

- (2) The function $Z_q(t)$ satisfies a functional equation

$$Z_q\left(\frac{1}{q^d t}\right) = \pm q^{de/2} t^e Z_q(t)$$

where e is the self-intersection of the diagonal in $X \times X$.

- (3) The Riemann hypothesis holds, i.e.

$$Z_q(t) = \frac{P_{1,q}(t) \cdots P_{2d-1,q}(t)}{P_{0,q}(t) \cdots P_{2d,q}(t)}$$

where $P_{0,q}(t) = 1 - t$, $P_{2d,q}(t) = 1 - q^d t$ and

$$P_{i,q}(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} t)$$

for $1 \leq i \leq 2d - 1$, and where the α_{ij} are algebraic integers of complex absolute value $|\alpha_{ij}| = q^{i/2}$.

- (4) Assume that X arises as the reduction of a variety defined over a number field K . Then the b_i are the topological Betti numbers of the complex variety $X_{\mathbb{C}}$ and $e = \sum_{i=0}^{2d} (-1)^i b_i$ is the Euler number.

The most difficult part of these conjectures is the Riemann hypothesis (in Subsection 2.3 we explain why this is related to the classical Riemann hypothesis), which was finally proved by Deligne [Del74] in 1974. These conjectures, especially point 4, establish a relationship between the arithmetic and the topology of an algebraic variety.

2.2. Étale cohomology and Frobenius. Deligne proved the Weil conjectures in the technical framework of étale cohomology.

This cohomology theory was developed by Grothendieck with a view towards proving the Weil conjectures. We refer the reader to [FK88] for an account of this theory, here we simply state its basic properties. Let Y be a smooth projective variety defined over an algebraically closed field of characteristic p , which can be either 0 or positive (we shall use this later for $Y = \overline{X}_p$ and $Y = \overline{X}$). Choose some prime $l \neq p$. Then there is a cohomology theory, called *étale cohomology*, which associates to Y certain \mathbb{Q}_l -vector spaces $H_{\text{ét}}^i(Y, \mathbb{Q}_l)$ for $i \geq 0$, which have many properties similar to the classical (singular) cohomology in characteristic 0. In particular, the étale cohomology groups have the following properties:

- (1) If Y has dimension d , then $H_{\text{ét}}^i(Y, \mathbb{Q}_l) = 0$ for $i > 2d$.
- (2) All \mathbb{Q}_l -vector spaces $H_{\text{ét}}^i(Y, \mathbb{Q}_l)$ are *finite dimensional*.
- (3) The cohomology groups behave *functorially* with respect to morphisms of smooth algebraic varieties.
- (4) *Poincaré duality* holds, i.e. there is an isomorphism $H_{\text{ét}}^{2d}(Y, \mathbb{Q}_l) \cong \mathbb{Q}_l$ and for each $i \leq d$ a perfect pairing

$$H_{\text{ét}}^i(Y, \mathbb{Q}_l) \times H_{\text{ét}}^{2d-i}(Y, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^{2d}(Y, \mathbb{Q}_l) \cong \mathbb{Q}_l.$$

- (5) The *Künneth decomposition* theorem for products holds.
- (6) The *Lefschetz fixed point formula* holds: Let $f : Y \rightarrow Y$ be a morphism such that the set of fixed points $\text{Fix}(f)$ is finite and $1 - df$ is injective (here

df denotes the differential of f), then

$$\# \text{Fix}(f) = \sum_{i=0}^{2d} (-1)^i \text{tr}(f^* | H_{\text{ét}}^i(Y, \mathbb{Q}_l)).$$

- (7) A *comparison theorem* with singular cohomology holds: if Y is smooth and projective over \mathbb{C} , then there are isomorphisms of \mathbb{C} -vector spaces

$$H_{\text{ét}}^i(Y, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C} \cong H^i(Y, \mathbb{C}).$$

The crucial property for our purposes is the Lefschetz fixed point formula. The condition that $1 - df$ is injective means that the fixed points have multiplicity one. Assume now that X is a smooth projective variety defined over \mathbb{Q} and that p is a prime of good reduction. Consider the geometric Frobenius morphism $\text{Fr}_p : \overline{X}_p \rightarrow \overline{X}_p$ given by sending the coordinates to their p -th power. Then

$$X(\mathbb{F}_{p^r}) = \text{Fix}(\text{Fr}_p^r).$$

Now choose a different prime $l \neq p$ (the choice of this prime will not matter). Then by the Lefschetz fixed point formula

$$N_{p^r} = \#X(\mathbb{F}_{p^r}) = \sum_{i=0}^{2d} (-1)^i \text{tr}((\text{Fr}_p^r)^* | H_{\text{ét}}^i(\overline{X}_p, \mathbb{Q}_l)).$$

We define polynomials

$$(1) \quad P_{i,p^r}(t) = \det(1 - (\text{Fr}_p^r)^* t | H_{\text{ét}}^i(\overline{X}_p, \mathbb{Q}_l)).$$

Using the identity

$$\sum_{n \geq 1} \text{tr}(f^n | V) \frac{t^n}{n} = -\log \det(1 - ft)$$

for an endomorphism f of a finite-dimensional vector space V , it is then essentially an exercise in formal power series to show that

$$Z_{p^r}(t) = \frac{P_{1,p^r}(t) \cdots P_{2d-1,p^r}(t)}{P_{0,p^r}(t) \cdots P_{2d,p^r}(t)}.$$

In particular, the polynomials $P_{i,p^r}(t)$ introduced in Section 2.1 coincide with the polynomials from equation (1).

Deligne's proof of the Riemann hypothesis for function fields of varieties defined over finite fields then follows from

Theorem 2.2 (Deligne, [Del74], [Del80]). *Let Y be a smooth projective variety defined over the field \mathbb{F}_q , where $q = p^r$ is a prime power. For all i and $l \neq p$ the eigenvalues of $\text{Fr}_q^* = (\text{Fr}_p^r)^*$ on $H_{\text{ét}}^i(\overline{Y}_p, \mathbb{Q}_l)$ are algebraic integers of absolute value $q^{i/2}$.*

In the case of a curve X there is another, more explicit, construction of the étale cohomology. Let $J(X)$ denote the Jacobian of X , then

$$H_{\text{ét}}^1(X, \mathbb{Q}_l) \cong \left(\lim_{\leftarrow n} J(X)[l^n] \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

If X is defined over \mathbb{Q} then there is a natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $J(\overline{X})[l^n]$ which yields a Galois action on $H_{\text{ét}}^1(\overline{X}, \mathbb{Q}_l)$. For a general variety X defined over \mathbb{Q} there is a natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H_{\text{ét}}^i(\overline{X}, \mathbb{Q})$, but this nice description is missing.

2.3. L -functions and Galois representations. We are now in a position to discuss L -functions. Let X be a model of a variety defined over \mathbb{Q} . Whereas we have, so far, considered the reduction of a variety X defined over \mathbb{Q} for a fixed prime p , we shall now vary the prime p . For any $i \geq 0$ we define the L -function

$$L(H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l), s) = (*) \prod_{p \in \mathcal{P}} \frac{1}{P_{i,p}(p^{-s})},$$

where \mathcal{P} is the set of primes of good reduction and $(*)$ denotes suitable Euler factors for the primes of bad reduction. In this survey article we shall not discuss the (difficult) question of how to define the Euler factors for bad primes in general, though we will explain this for the case of elliptic curves (see Section 4.1). Note that if $X = \mathbb{P}^1$ and $i = 0$, then we obtain exactly Riemann's zeta function $\zeta(s)$, and if X is a smooth projective curve, one thus obtains the Hasse zeta function of X .

In most cases, the interesting cohomology of a variety X is the middle cohomology. For this reason we define the L -series of a d -dimensional variety X as

$$L(X, s) := L(H_{\text{ét}}^d(\overline{X}, \mathbb{Q}_l), s).$$

We have already remarked that $H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C} \cong H^i(X_{\mathbb{C}}, \mathbb{C})$ as \mathbb{C} -vector spaces.

Let p be a good prime. A further comparison theorem says that the reduction map gives an isomorphism

$$\psi : H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l) \xrightarrow{\sim} H_{\text{ét}}^i(\overline{X}_p, \mathbb{Q}_l)$$

as \mathbb{Q}_l -vector spaces. The isomorphism ψ is also compatible with the Galois module structure of these spaces in the following sense. The Galois group $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts on \overline{X}_p thus making $H_{\text{ét}}^i(\overline{X}_p, \mathbb{Q}_l)$ a $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ -module. Similarly, the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on \overline{X} giving $H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l)$ a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module structure. This makes Poincaré duality a Galois-equivariant pairing.

For every embedding $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ one obtains an embedding

$$\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Two different embeddings yield conjugated subgroups of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\tilde{\text{Fr}}_p$ be a lift of Fr_p . Then the action of $\psi \circ \tilde{\text{Fr}}_p$ on $H_{\text{ét}}^i(\overline{X}_p, \mathbb{Q}_l)$ is conjugated to the action of Fr_p .

Let

$$\rho_i(X) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l)).$$

be the natural Galois representations. Since any two lifts of Fr_p are conjugate they have the same trace and determinant. In this way we can still speak about the trace and determinant of Frobenius at a good prime p in connection with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representations.

3. AUTOMORPHIC ORIGIN OF l -ADIC REPRESENTATIONS

3.1. The Galois representation associated to a modular form. To recall the notion of (elliptic) modular form, we recall first that the group $\text{SL}(2, \mathbb{Z})$ acts on the complex upper half plane \mathbb{H} by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

We can define \mathbb{H}^* by adding to \mathbb{H} a copy of $\mathbb{P}^1(\mathbb{Q})$ in the following way: the point $(x : 1)$ is identified with $x \in \mathbb{Q} \subset \mathbb{C}$ and the point $(1 : 0)$ is identified with the point at infinity along the complex axis, usually denoted by $i\infty$. The action of $\mathrm{SL}(2, \mathbb{Z})$ extends to \mathbb{H}^* .

For a given integer N we define the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}); c \equiv 0 \pmod{N} \right\}.$$

The quotient $Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$ is a Riemann surface, which can be compactified to a projective curve $X_0(N) = \Gamma_0(N) \backslash \mathbb{H}^*$. The curve $X_0(N)$ is called the *modular curve of level N* and the points in $X_0(N) \setminus Y_0(N)$ are called *cusps*.

An elliptic modular form of *weight k* and *level N* is a holomorphic function

$$f : \mathbb{H} \rightarrow \mathbb{C}$$

which satisfies the following properties:

- (1) For matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ the function f transforms as

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

- (2) The function f is holomorphic at the cusps.

To explain the latter condition, we consider one of the cusps, namely the cusp given by adding $i\infty$. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is an element of $\Gamma_0(N)$ and we thus have

$$f(\tau + 1) = f(\tau).$$

Writing $q = \exp^{2\pi i\tau}$ we can, therefore, consider the Fourier expansion

$$f(\tau) = f(q) = \sum_n b_n q^n.$$

The cusp $i\infty$ corresponds to $q = 0$ and f is holomorphic at the cusp $i\infty$ if $b_n = 0$ for $n < 0$. We say that f vanishes at this cusp, if in addition $b_0 = 0$. A *cusp form* is a modular form which vanishes at all cusps. We shall also consider modular forms with a mod N -Dirichlet character χ . In this case the transformation behaviour given in the definition of modular forms is

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau).$$

The vector space of weight k cusp forms for $\Gamma_0(N)$ is denoted by $S_k(\Gamma_0(N))$ and, similarly, the vector space of weight k cusp forms with character χ is denoted by $S_k(\Gamma_0(N), \chi)$.

On $S_k(\Gamma_0(N), \chi)$ we can define operators T_p for prime numbers $p \nmid N$. For a precise definition we refer to [Kna92, page 244]. The operators T_p are called *Hecke operators*. They generate a subalgebra of $\mathrm{End}(S_k(\Gamma_0(N), \chi))$ called the *Hecke algebra* \mathbb{T} .

Since Hecke operators T_q and T_p commute for distinct primes p, q not dividing N , we have simultaneous eigenspaces for all T_p . A form that is an eigenvector for

all T_p is called a *Hecke eigenform*. Let $g \in S_k(\Gamma_0(N), \chi)$ be a Hecke eigenform, so $T_p(g) = c_p g$. Then there is a minimal $N' \mid N$ and a unique eigenform

$$f = \sum b_n q^n \in S_k(\Gamma_0(N'), \chi)$$

with the same system of eigenvalues such that $b_p = c_p$ for all $p \nmid N$. The eigenform f is called a (normalised) newform.

Let f be a normalised Hecke eigenform of some level N and weight $k > 1$. We will now explain how one can associate to f a Galois representation

$$\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(2, \mathbb{Q}_l)$$

in the case that $k > 2$. The case $k = 2$ is treated in Remark 3.1. The following construction is due to Deligne [Del71]. Using the trivial inclusion $S_k(\Gamma_0(N), \chi) \subset S_k(\Gamma_0(NM), \chi)$ we may assume that $N > 4$.

In order to define ρ_f we start by considering the subgroup $\Gamma_1(N) \subset \Gamma_0(N)$ of matrices M such that

$$M \equiv \begin{pmatrix} 1 & \bar{b} \\ 0 & 1 \end{pmatrix} \pmod{N}.$$

A modular form in $S_k(\Gamma_0(N), \chi)$ is also a modular form for the group $\Gamma_1(N)$. Note, however, that the construction we present below uses the group $\Gamma_1(N)$ in an essential way. As above, we can define modular curves $Y_1(N)$ and $X_1(N)$. One can show that both curves have a natural model over \mathbb{Q} . Since we assumed that $N > 4$ we have that $Y_1(N)$ is a *fine* moduli-space for pairs (E, P) with E an elliptic curve and P a point of order N . Let $\pi : \mathcal{E}_1(N) \rightarrow Y_1(N)$ be the associated universal family. We define the l -adic sheaf $\mathcal{F}_k = \text{Sym}^{k-2} R^1 \pi_* \mathbb{Q}_l$ on the étale site of $Y_1(N)$.

Let $j : Y_1(N) \rightarrow X_1(N)$ be the natural morphism. Then the étale cohomology group

$$H := H_{\text{ét}}^1(\overline{X_1(N)}, j_* \mathcal{F}_k)^\vee$$

has an action of the Hecke-algebra $\mathbb{T} \otimes_{\mathbb{Q}} \mathbb{Q}_l$. It turns out that H is a direct sum of free rank two Hecke-algebras. Recall that we had fixed a normalised Hecke eigenform f . Let b_p denote the eigenvalue of T_p . Consider the morphism $\mathbb{T} \otimes_{\mathbb{Q}} \mathbb{Q}_l \rightarrow \overline{\mathbb{Q}_l}$ defined by $T_p \rightarrow b_p$. Define

$$V_f := H_{\text{ét}}^1(\overline{X_1(N)}, j_* \mathcal{F}_k)^\vee \otimes_{\mathbb{T} \otimes_{\mathbb{Q}} \mathbb{Q}_l} \overline{\mathbb{Q}_l}.$$

It turns out that V_f is two-dimensional and that there is a natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on V_f . This gives our representation ρ_f .

Remark 3.1. If f has weight $k = 2$ then $f \frac{dq}{q}$ induces a holomorphic one-form on $X_1(N)$. Consider the two-dimensional subspace

$$V'_f := f \frac{dq}{q} \mathbb{C} \oplus \overline{f \frac{dq}{q}} \mathbb{C} \subset H^1(\overline{X_1(N)}, \mathbb{C})$$

Using the comparison theory of étale cohomology and singular cohomology we obtain a two-dimensional subspace V_f of $H_{\text{ét}}^1(\overline{X_1(N)}, \overline{\mathbb{Q}_l})$. It turns out that V_f is Galois-invariant. One can show that this gives a definition of ρ_f .

Similarly, if f has weight $k > 2$ then $f \frac{dq_1}{q_1} \wedge \cdots \wedge \frac{dq_{k-2}}{q_{k-2}}$ induces a holomorphic $k - 2$ -form on $\text{Sym}^{k-2} \overline{\mathcal{E}_1(N)}$. This leads to a two-dimension subrepresentation of $H^{k-2}(\overline{\mathcal{E}_1(N)}, \overline{\mathbb{Q}_l})$. This provides an alternative definition of ρ_f . Although this

construction seems to be known to many experts, we could only find references for the special cases $k = 3$ [Shi72] and $k = 4$ [SY01].

To a normalised Hecke newform f we can associate an L -function $L(f, s)$. One way of defining this is $L(f, s) := L(\rho_f, s)$, i.e., as the L -series of the Galois-representation ρ_f . Equivalently, we can do the following: consider the Fourier expansion $f = \sum b_n q^n$. One can associate to this its *Mellin transform*

$$L(f, s) = \sum_n b_n n^{-s}.$$

If f is a normalised Hecke newform with respect to the group $\Gamma_0(N)$, then the Fourier coefficients satisfy the properties

$$\begin{aligned} b_{p^r} b_p &= b_{p^{r+1}} + p^{k-1} b_{p^{r-1}} \text{ for } p \text{ prime, } p \nmid N \\ b_{p^r} &= (b_p)^r \text{ for } p \text{ prime, } p \mid N \\ b_n b_m &= b_{nm} \text{ if } (n, m) = 1 \end{aligned}$$

where k is the weight of the form f . It follows from this that the series $L(f, s)$ has a product expansion

$$(2) \quad L(f, s) = \sum_{n \geq 1} b_n n^{-s} = \prod_p \frac{1}{1 - b_p p^{-s} + \chi(p) p^{k-1-2s}}$$

where $\chi(p) = 0$ if $p \mid N$.

3.2. Fontaine-Mazur conjecture. Here we shall briefly discuss the Fontaine-Mazur conjecture. A more detailed discussion of this conjecture as well as many references for the facts mentioned below, can be found in Taylor's survey paper [Tay04].

Consider a variety X/\mathbb{Q} . We have already mentioned that the étale cohomology groups $H_{\text{ét}}^i(\overline{X}, \overline{\mathbb{Q}}_l)$ have a natural Galois action, i.e., there is a continuous homomorphism

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(H_{\text{ét}}^i(\overline{X}, \overline{\mathbb{Q}}_l)).$$

In order to discuss Galois-representations we first have to list some subgroups of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ we obtain $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a subgroup of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Two such embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ give subgroups of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that differ only up to conjugacy. Fix for every prime p an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

The group $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ maps onto $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The kernel of this map is called the inertia subgroup at p , denoted by I_p .

Let V be a finite dimensional vector space over \mathbb{Q}_l . Let $\rho' : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(V)$ be an irreducible subrepresentation of ρ . Then the representation ρ' satisfies the following properties:

- (1) ρ' is only ramified at finitely many primes. I.e., $\rho'(I_p) = \{1\}$ for all but finitely many primes p .
- (2) The restriction of ρ' to the subgroup $\text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ is of de Rham type. (For a definition of this see [FO94a], [FO94b]. Note that this is quite restrictive.).
- (3) For all but finitely many primes p we have that all roots α of the characteristic polynomial of $\rho(\text{Fr}_p)$ on V satisfy $|\alpha| = p^{i/2}$, with $|\cdot|$ the complex absolute value.

In general, an irreducible l -adic Galois-representation

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(V)$$

with V a finite dimensional $\overline{\mathbb{Q}_l}$ vector space, is called *geometric* if it satisfies the above properties 1 and 2.

The i -th *Tate twist* of a Galois-representation ρ is defined as the Galois representation obtain by tensoring ρ with $\mu_{l^\infty}^{\otimes i}$ if $i > 0$ and tensoring ρ with $\mu_{l^\infty}^{V \otimes -i}$ if $i < 0$. Here μ_{l^n} denotes the l^n -th roots of unity and $\mu_{l^\infty} = \varprojlim \mu_{l^n}$.

Conjecture 3.2 (Fontaine-Mazur, [FM95, Conjecture 1]). *Let V be a finite dimensional $\overline{\mathbb{Q}_l}$ -vector space and let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(V)$ be an irreducible geometric l -adic representation. Then there exists a smooth projective variety X/\mathbb{Q} and an integer i such that ρ is a Tate twist of an irreducible subrepresentation of the standard Galois representation on $H^i(\overline{X}, \overline{\mathbb{Q}_l})$. In particular ρ satisfies property 3. Moreover, the L -function $L(\rho, s)$ has a meromorphic continuation to \mathbb{C} .*

A Tate twist of an irreducible subrepresentation of the standard Galois representation on $H^i(\overline{X}, \overline{\mathbb{Q}_l})$ is called a *Galois representation coming from geometry*.

In the two-dimensional case they make a stronger conjecture:

Conjecture 3.3 (Fontaine-Mazur, [FM95, Conjecture 3c]). *Let V be a two-dimensional $\overline{\mathbb{Q}_l}$ -vector space. Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(V)$ be an irreducible geometric Galois representation that is not a Tate twist of a finite representation. Then ρ is isomorphic to the Tate twist of a Galois representation associated to a modular form.*

Later on we will introduce rigid Calabi-Yau threefolds. A rigid Calabi-Yau threefold X has $b_3(X) = 2$. Suppose X is defined over \mathbb{Q} . Then we have a two-dimensional Galois representation. The above conjecture states that the Galois representation on $H_{\text{ét}}^3(\overline{X}, \overline{\mathbb{Q}_l})$ is coming from a modular form. This conjecture is also stated in [SY01].

3.3. Langlands' programme. Here we shall only give a very brief outline. For a more extended introduction to this subject we refer to [Tay04]. In the previous section we have seen a relation between Galois-representations, varieties over \mathbb{Q} and L -series. In this section we introduce a fourth class of objects, namely automorphic representations.

Let $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/N\mathbb{Z}$. Denote $\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ and $\mathbb{A} = \mathbb{A}^\infty \times \mathbb{R}$. As a ring \mathbb{A}^∞ is the subring $\prod_p \mathbb{Q}_p$ consisting of elements (x_p) such that $x_p \in \mathbb{Z}_p$ for all but finitely many primes p . Note that the topology of \mathbb{A}^∞ is different from the subspace topology of $\prod \mathbb{Q}_p$.

Let $W_{\mathbb{Q}_p}$ be the Weil-group of \mathbb{Q}_p , that is, all $\sigma \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that their image in $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is a power of Fr_p . This is a dense subgroup of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Local class field theory gives a 'natural' isomorphism

$$\text{Art}_p : \mathbb{Q}_p^* \xrightarrow{\sim} W_{\mathbb{Q}_p}^{\text{ab}},$$

where $*^{\text{ab}}$ means the abelianised group. Furthermore, global class field theory gives an isomorphism

$$\text{Art} : \mathbb{A}^*/\mathbb{Q}^*\mathbb{R}_{>0}^* \xrightarrow{\sim} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\text{ab}}$$

which is compatible with Art_p . (A large part of) class field theory can be described in terms of the map Art . The Langlands' programme can be considered as an attempt to generalise this to $\text{GL}(n, \mathbb{Q}) \backslash \text{GL}(n, \mathbb{A})$.

Fix $n \geq 1$. An automorphic form is a smooth function

$$\text{GL}(n, \mathbb{Q}) \backslash \text{GL}(n, \mathbb{A}) \rightarrow \mathbb{C}$$

satisfying certain conditions: for example, the translates of f under $\text{GL}(n, \hat{\mathbb{Z}}) \times O(n)$ (a maximal compact subgroup of $\text{GL}(n, \mathbb{A})$) form a finite-dimensional vector space. Other conditions concern a generalisation of the vanishing of f at cusps and limit the growth of f . For reasons of space we will not give a complete definition, see [Tay04]. To an automorphic form one can associate an infinitesimal character H , where H is a multiset of n complex numbers (i.e., an element of $\text{Sym}^n \mathbb{C}$). Fix such a multiset H . Let $\mathcal{A} := \mathcal{A}_H^o(\text{GL}(n, \mathbb{Q}) \backslash \text{GL}(n, \mathbb{A}))$ be the vector space of cuspidal automorphic forms with infinitesimal character H . We would like to consider \mathcal{A} as a representation of $\text{GL}(n, \mathbb{A})$. It turns out that $\text{GL}(n, \mathbb{R})$ does not fix \mathcal{A} . However, \mathcal{A} has an action of $\text{GL}(n, \mathbb{A}^\infty) \times O(n)$ and an action of the Lie-algebra \mathfrak{gl}_n . An irreducible constituent of \mathcal{A} is called a *cuspidal automorphic representation* of $\text{GL}(n, \mathbb{A})$.

Consider $n = 1$. One easily describes $\mathcal{A}_{\{s\}}^o(\mathbb{Q}^* \backslash \mathbb{A}^*)$, with $s \in \mathbb{C}$, in terms of $\mathcal{A}_{\{0\}}^o(\mathbb{Q}^* \backslash \mathbb{A}^*)$. It turns out that elements in this space are locally constant functions f on

$$\hat{\mathbb{Z}}^* \xrightarrow{\sim} \mathbb{A}^* / \mathbb{Q}^* \mathbb{R}_{>0}^*.$$

Thus $\mathcal{A}_{\{0\}}^o(\mathbb{Q}^* \backslash \mathbb{A}^*)$ is generated by continuous characters $\hat{\mathbb{Z}}^* \rightarrow \mathbb{C}$.

Consider $n = 2$. Write $H = \{s, t\}$. Then $\mathcal{A} \neq (0)$ only if $s - t \in i\mathbb{R}$, $s - t \in \mathbb{Z}$ or $s - t \in (-1, 1)$. If $s - t \in \mathbb{Z}_{>0}$ then to every element in \mathcal{A} we can associate a weight $1 + s - t$ modular form and one can determine its level in terms of \mathcal{A} . This shows that automorphic forms are generalisations of modular forms.

Suppose $n > 1$. As in the case of modular forms we can associate local L -series $L_p(\pi, s)$ to an automorphic form π . (Basically, we restrict the automorphic form π to $\pi_p : \text{GL}(n, \mathbb{Q}_p) \subset \text{GL}(n, \mathbb{A})$ in order to define a local L -factor $L_p(\pi_p, s)$.) The Euler product $\prod_p L_p$ converges on some half-plane, and has a meromorphic continuation to \mathbb{C} .

A weak form of the Langlands' programme is to prove that every L -series of an automorphic form is also the L -series of an irreducible Galois-representation coming from geometry and that this correspondence is 'natural'.

4. TWO-DIMENSIONAL GALOIS REPRESENTATIONS

Wiles' proof of a special case of the Taniyama-Shimura-Weil conjecture was the essential ingredient in his proof of Fermat's Last Theorem. In this section we briefly recall his result and discuss generalisations to higher dimension, in particular to rigid Calabi-Yau threefolds.

4.1. Elliptic curves and Wiles' theorem. We consider elliptic curves over \mathbb{Q} , which we can assume to be in Weierstrass form

$$y^2 = x^3 + Ax + B, \quad \Delta = 4A^3 + 27B^2 \neq 0$$

where $A, B \in \mathbb{Q}$. For good primes p the Lefschetz fixed point formula reads

$$N_p = \#E(\mathbb{F}_p) = 1 - \text{tr}(\text{Fr}_p^* | H_{\text{ét}}^1(\overline{E}_p, \mathbb{Q}_l)) + p.$$

Writing

$$a_p = \text{tr}(\text{Fr}_p^* | H_{\text{ét}}^1(\overline{E}_p, \mathbb{Q}_l))$$

this formula simply becomes

$$N_p = 1 - a_p + p.$$

The Taniyama-Shimura-Weil conjecture (TSW), in one of its forms, predicts that the sequence N_p , resp. a_p , is given (for good primes) by the Fourier coefficients of a modular form. More precisely,

Theorem 4.1 (Wiles-Taylor-Breuil-Conrad-Diamond, [BCDT01]). *Let E be an elliptic curve defined over \mathbb{Q} . Then E is modular, i.e., there exists a Hecke newform of weight two with Fourier expansion $f(q) = \sum_n b_n q^n$ such that for all primes p of good reduction*

$$a_p = 1 - N_p + p = b_p.$$

Before commenting further on this theorem, we want to discuss an example. Let E be the elliptic curve given by the equation

$$y^2 = x^3 + 1.$$

Then we find the following values for N_p and a_p :

p	2	3	5	7	11	13	17	19
N_p	3	4	6	12	12	12	18	12
a_p	0	0	0	-4	0	2	0	8

In this case it is easy to describe the associated modular form explicitly. Recall that the Dedekind η -function is defined by

$$\eta(\tau) = \eta(q) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n).$$

Then the form

$$f(\tau) = \eta(6\tau)^4 = q \prod_{n \geq 1} (1 - q^{6n})^4 = 1 - 4q^7 + 2q^{13} + 8q^{19} + \dots$$

is a weight two form of level 36 and one can easily verify the condition $a_p = b_p$ for any primes within one's computing capacity. One can provide a complete proof for this fact using Theorem 4.3 or the theory of CM-elliptic curves and CM-modular forms.

Some comments regarding Theorem 4.1 are in order at this point. The Taniyama-Shimura-Weil conjecture also makes a prediction about the level of the modular form. If the equation of E is a *global minimal Weierstrass equation* (this is a Weierstrass equation for E whose discriminant Δ is divisible by a minimal power of p for all primes p , see [Kna92, Chapter X.1]), then the level of f is only divisible by the primes of bad reduction. More precisely, if $p \neq 2, 3$, then the level is at most divisible by p^2 , for $p = 3$ it is at most divisible by p^5 and for $p = 2$ by p^8 (see [Ogg67]). In fact, the level equals the conductor of E (see the introduction of [BCDT01]).

Also, the statement that E is modular, can be expressed in different, though equivalent, ways. One of these is to say that an elliptic curve E is modular, if there exists a non-constant morphism $X_0(N) \rightarrow E$ defined over the rationals. This characterisation has no satisfactory known counterpart in the higher dimensional case.

Yet another equivalent formulation, and one that can be generalised, is to say that the L -series of E is that of a weight two form f . In Section 2.3 we have already defined an L -function for varieties defined over the \mathbb{Q} . In the case at hand we can make this explicit, including a definition of the Euler factors associated to the bad primes: we first assume that the elliptic curve E is given by a global minimal Weierstrass equation. Let $a_p = 1 - N_p + p$ for good primes and define

$$a_p = \begin{cases} 1 & \text{in the case of split multiplicative reduction} \\ -1 & \text{in the case of non-split multiplicative reduction} \\ 0 & \text{in the case of additive reduction.} \end{cases}$$

Then we set

$$L(E, s) = \prod_{p|\Delta} \frac{1}{1 - a_p p^{-s}} \prod_{p \nmid \Delta} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$

In Section 3.1 we have also introduced the L -series of a normalised Hecke eigenform. Modularity of E can then be rephrased as an equality of L -series

$$L(E, s) = L(f, s)$$

for a suitable Hecke eigenform of weight two.

4.2. The theorem of Dieulefait and Manoharmayum. In view of Wiles' result it is natural to ask for generalisations to other classes of varieties. Probably the most natural class to be considered for this question is that of Calabi-Yau varieties. A smooth projective variety is called a *Calabi-Yau* variety if the canonical bundle is trivial, i.e.,

$$(3) \quad \omega_X = \mathcal{O}_X$$

and the following vanishing holds

$$(4) \quad h^i(X, \mathcal{O}_X) = 0 \text{ for } 0 < i < \dim X.$$

For a curve, condition (4) is empty and we obtain the class of elliptic curves. Calabi-Yau varieties of dimension two are K3 surfaces (see Section 5.1 for a discussion of this case). Calabi-Yau threefolds have been investigated very thoroughly. A lot of this interest comes from the fact that they play an essential role in string theory. The easiest examples of Calabi-Yau threefolds are quintic hypersurfaces in \mathbb{P}^4 and complete intersections of type (2, 4) or (3, 3) in \mathbb{P}^5 . For a discussion of Calabi-Yau varieties, which appear as complete intersections in toric varieties, and their connection with mirror symmetry we refer the reader to [Bat94], [BB96].

If we look at Calabi-Yau varieties from the point of view of modularity, then it is natural to start with examples whose middle cohomology is as simple as possible. By definition, we always have $h^{3,0}(X) = h^{0,3}(X) = 1$. A Calabi-Yau variety X is called *rigid* if $h^{2,1}(X) = h^{1,2}(X) = 0$. The name comes from the fact that this condition implies $H^1(X, T_X) = 0$, and hence that X has no infinitesimal complex deformations. In this case the middle cohomology

$$H^3(X) = H^{3,0}(X) \oplus H^{0,3}(X) \cong \mathbb{C}^2$$

is two-dimensional.

Now assume that X is a rigid Calabi-Yau manifold, which is defined over \mathbb{Q} . Then one can show that the determinant of Frobenius on $H_{\text{ét}}^3(\overline{X}, \mathbb{Q}_l)$ is p^3 . Hence

$$P_{3,p}(t) = \det(1 - t \text{Fr}_p^* | H_{\text{ét}}^3(\overline{X}, \mathbb{Q}_l)) = 1 - \text{tr}((\text{Fr}_p)^* | H_{\text{ét}}^3(\overline{X}, \mathbb{Q}_l))t + p^3 t^2.$$

We shall again set

$$a_p = \text{tr}(\text{Fr}_p^* | H_{\text{ét}}^3(\overline{X}, \mathbb{Q}_l)).$$

The L -series of (the middle cohomology of) X is then of the form

$$L(X, s) = (*) \prod_{p \in \mathcal{P}} \frac{1}{1 - a_p p^{-s} + p^{3-2s}}$$

where the factor $(*)$ denotes the Euler factors associated to the bad primes, and \mathcal{P} is the product of good primes. Comparing this to equation (2) leads one to expect a relationship with a modular form of weight four. Indeed, modularity of rigid Calabi-Yau threefolds defined over \mathbb{Q} , had been conjectured for some time. This can be seen as a special form of the Fontaine-Mazur conjecture [FM95] (see Subsection 3.2) and has been stated explicitly by Saito and Yui [SY01]. In recent years, many examples of geometrically interesting rigid Calabi-Yau varieties have been found and in all known cases it has been possible to determine the associated weight four cusp form (see also Section 4.3).

A general modularity result has been proved by Dieulefait and Manoharmayum [DM03], with further improvements provided later by Dieulefait [Di04b], [Di05].

Theorem 4.2. *Let X be a rigid Calabi-Yau threefold defined over \mathbb{Q} , and assume that one of the following conditions holds:*

- (1) *X has good reduction at 3 and 7 or*
- (2) *X has good reduction at 5 or*
- (3) *X has good reduction at 3 and the trace of Fr_3 on $H_{\text{ét}}^3(\overline{X}, \mathbb{Q}_l)$ is not divisible by 3.*

Then X is modular. More precisely

$$L(X, s) \doteq L(f, s)$$

for some weight four modular form, where \doteq means equality up to finitely many Euler factors.

Very recently Dieulefait has informed us that the above theorem can be further improved. More precisely, he can show that X is modular if it has good reduction at 3. The proof goes along the lines of [Di05]. This requires that a modularity lifting result of Diamond, Flach and Guo can be extended to the case of weight 4 and characteristic 3. Indeed, Kisin [Kis05] has recently announced such an extension.

We remark that, as in the case of elliptic curves, one expects that the level of the modular form is only divisible by the primes of bad reduction. For the question of determining the level of f see also [Di04a] and Section 7. Very little is known about the Euler factors associated to the primes of bad reduction.

The above result is purely an existence result and does not provide a method to determine the form f explicitly. There is, however, a method due to Faltings, Serre and Livné, which is very effective, if one wants to prove that a candidate modular form is indeed the right one. We shall discuss this method in the next section.

Finally, we would like to remark that a number of non-rigid Calabi-Yau varieties have been found, for which modularity has been established. Most of the examples known up to date are of two types. They are either of Kummer type or they contain (many) elliptic ruled surfaces (or both). In both cases the geometry of the variety is such that the middle cohomology breaks up into two-dimensional

pieces. The first type of example was found by Livné and Yui [LY03], for Calabi-Yau varieties containing elliptic ruled surfaces see [HV03] and [HV05a]. We shall discuss examples, mostly rigid, in more detail in Section 5.

4.3. The method of Faltings-Serre-Livné. Given a rigid Calabi-Yau threefold, defined over \mathbb{Q} , one often wants to determine the associated modular form explicitly. Generally, it is not hard to find a suitable candidate for such a form f : counting points, one can determine the first few Fourier coefficients of f and then compare these to existing lists of Hecke eigenforms such as [St]. In order to actually prove that the L -series of the variety and that of the modular form coincide, one can try to establish the equality of (the semi-simplifications of) the corresponding two-dimensional Galois representations (over \mathbb{Q}_2). In order to prove that two Galois representations ρ_1, ρ_2 have isomorphic semi-simplifications, it suffices to prove that for all primes p in a certain finite set of primes T the traces of $\rho_1(\text{Fr}_p)$ and $\rho_2(\text{Fr}_p)$ are equal. This result is due to Faltings [Fal83]. In the sequel one of the representations is the Galois representation on the cohomology, the other representation is of the form ρ_f , with f a modular form. It is known that ρ_f is already simple, see [Rib77, Theorem 2.3].

If the representations have even traces then one can determine the set T effectively. This method is due to Serre and was recast by Livné [Liv87]. Otherwise one can use an approach of Serre [Ser85], which, however can only be made explicit in special cases.

Theorem 4.3. *Let ρ_1, ρ_2 be two continuous two-dimensional two-adic representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, unramified outside a finite set S of prime numbers. Let \mathbb{Q}_S be the compositum of all quadratic extensions of \mathbb{Q} , which are unramified outside S and let T be a set of primes, disjoint from S , such that $\text{Gal}(\mathbb{Q}_S/\mathbb{Q}) = \{\text{Fr}_p \mid p \in T\}$. Suppose that*

- (1) $\text{tr } \rho_1(\text{Fr}_p) = \text{tr } \rho_2(\text{Fr}_p)$ for all $p \in T$,
- (2) $\det \rho_1(\text{Fr}_p) = \det \rho_2(\text{Fr}_p)$ for all $p \in T$,
- (3) $\text{tr } \rho_1 \equiv \text{tr } \rho_2 \equiv 0 \pmod{2}$ and $\det \rho_1 \equiv \det \rho_2 \pmod{2}$.

Then ρ_1 and ρ_2 have isomorphic semisimplifications, and hence $L(\rho_1, s) \doteq L(\rho_2, s)$. In particular, the good Euler factors of ρ_1 and ρ_2 coincide.

The gist of this theorem is, that, provided the parity of trace and determinant coincide, it is sufficient to check only the equality of the traces and determinants for a *finite* number of explicitly known primes, in order to establish that the semi-simplifications are isomorphic.

To check the parity of trace and determinant is in practice often easy. The main point is to determine a suitable set T of primes. Given this, the calculation of the traces a_p for $p \in T$ is often straightforward. Of course, the set T depends on the bad primes of X . But in most concretely given examples, the set T is quite small.

If the traces are not even, then one can apply Serre's approach, provided that the mod 2-reductions $\bar{\rho}_1$ and $\bar{\rho}_2$ are absolutely irreducible and isomorphic. This means that their kernels cut out the same S_3 -extension K of \mathbb{Q} . In this situation, one has to compute all extensions L of K such that $\text{Gal}(L/\mathbb{Q})$ is either S_4 or $S_3 \times C_2$ and which are unramified outside the ramification loci of the representations ρ_i . Let T be a set of primes such that for any such extension L there is a prime $p \in T$ with Fr_p of maximal order in $\text{Gal}(L/\mathbb{Q})$. Then, proving the isomorphism of the Galois representations amounts to checking equality of the traces at the primes of T .

In practice, this method can only be used if there are only a few small bad primes, since the available tables of such number fields are limited (cf. [Jon]). We will sketch this approach for the Schoen quintic in Section 5.2.1. In the context of non-rigid Calabi threefolds, applications of this method can also be found in [Schu04b].

5. EXAMPLES

In this chapter we shall discuss various examples which have been found in recent years. For an extensive survey of modular Calabi-Yau threefolds we refer the reader to the book of Meyer [Mey05].

5.1. K3 surfaces. In Sections 4.1 and 4.2 we discussed elliptic curves and (rigid) Calabi-Yau threefolds. The reader may ask why we did not consider K3 surfaces before proceeding to dimension three. The reason is that in even dimension the middle cohomology will always contain algebraic classes, which changes the situation. If S is any K3 surface, then the middle cohomology has dimension 22, more precisely

$$H^2(S, \mathbb{Z}) = 3U + 2E_8(-1)$$

where U denotes the hyperbolic plane, i.e. the free lattice of rank 2, equipped with the intersection form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the unique positive definite, even, unimodular lattice of rank 8. The notation $E_8(-1)$ indicates that we take the negative form. The Néron-Severi group of S is the group of divisors modulo numerical equivalence or, equivalently,

$$\text{NS}(S) = H^{1,1}(S) \cap H^2(S, \mathbb{Z}).$$

Its rank $\rho(S)$ is called the *Picard number* of S . This can vary (in characteristic 0) from 0, in which case the K3 surface is not algebraic, to 20. K3 surfaces S with maximal Picard number $\rho(S) = 20$ are called *singular* K3 surfaces (a common but misleading terminology which does not mean that the surface has singularities). Singular K3 surfaces have no moduli, but they form an everywhere dense set (in the analytic topology) in the period domain of K3 surfaces. The orthogonal complement of $\text{NS}(S)$ in $H^2(S, \mathbb{Z})$ is denoted by $T(S)$ and is called the *transcendental lattice* of the surface S . If S is a singular K3 surface, this is a positive definite even lattice of rank 2. Shioda and Inose have proved that the map which associates to a singular K3 surface S the oriented lattice $T(S)$ (where the orientation is given by the 2-form on S) defines a bijection between the sets of singular K3 surfaces modulo isomorphism on the one hand and positive definite, integral binary quadratic forms modulo $\text{SL}(2, \mathbb{Z})$ on the other hand [SI77]. One also refers to the discriminant of the lattice $T(S)$ as the *discriminant* of the surface S . The injectivity of the above map is a consequence of the Torelli theorem. To prove surjectivity Shioda and Inose proceed as follows. Fix a positive definite even lattice of rank 2 and discriminant $d > 0$. Recall that such a lattice determines uniquely a *reduced* quadratic form

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where $\det(Q) = d$ and $a > 0, -a < b \leq a \leq c$ (and $b \geq 0$ if $a = c$). Set

$$\tau = \frac{-b + \sqrt{-d}}{2a} \text{ and } \tau' = \frac{b + \sqrt{-d}}{2}.$$

By construction, the elliptic curves E_τ and $E_{\tau'}$ have complex multiplication and are isogenous. The abelian surface $A = E_\tau \times E_{\tau'}$ has Picard number $\rho(A) = 4$ and the intersection form on its transcendental lattice $T(A)$ is given by Q , see [SM74]. For the Kummer surface $\text{Km}(A)$ one has $T(\text{Km}(A)) = T(A)(2)$, i.e. the form defined by Q , but multiplied by 2. In order to find a K3 surface with the form given by Q , Shioda and Inose exhibit an elliptic fibration on $\text{Km}(A)$ and construct a suitable double cover, resulting in another K3 surface S for which $T(S) = T(A)$. The surfaces S and $\text{Km}(A)$ are related by a *Nikulin involution*. More precisely, S admits an involution ι with exactly 8 fixed points, leaving the non-degenerate 2-form on S invariant, such that $\text{Km}(A)$ is the minimal resolution of the quotient S/ι .

We shall now turn to the question of modularity of K3 surfaces. Note that we are now mostly considering varieties defined over a number field K rather than over \mathbb{Q} . We can, however, still define an L -series for a variety X defined over K . In order to do this, we define for every prime ideal $\mathfrak{p} \in \text{Spec } \mathcal{O}_K$ and each integer i the polynomial

$$P_{i,\mathfrak{p}}(t) = \det(1 - t \text{Fr}_{\mathfrak{p}}^* | H_{\text{ét}}^i(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_l)).$$

If the dimension of X is d , we set

$$L(X/K, s) = (*) \prod_{\mathfrak{p} \in \mathcal{P}} \frac{1}{P_{d,\mathfrak{p}}((\text{Nm } \mathfrak{p})^{-s})},$$

where Nm denotes the norm endomorphism from K to \mathbb{Q} and \mathcal{P} is the set of primes of good reduction. The symbol $(*)$ stands for suitably defined Euler factors at the bad primes. In the case $K = \mathbb{Q}$ this coincides with our previous definition.

Since very little is known about the modularity of K3 surfaces which are not singular, we shall restrict ourselves to the singular case. Recall that a modular form f with Fourier expansion $f = \sum_n a_n q^n$ has complex multiplication by a Dirichlet character χ if $f = f \otimes \chi$ where

$$f \otimes \chi = \sum_n \chi(n) a_n q^n.$$

The character χ is necessarily quadratic. We say that f has CM by the imaginary quadratic field K , if f has CM by the corresponding quadratic character. CM forms are closely related to Hecke Grössencharacters: by a result of Ribet, every newform with CM comes from a Grössencharacter of an imaginary quadratic field K . For a treatment of the connection between CM forms and Hecke Grössencharacters see [Rib77].

Let S be a singular K3 surface of discriminant d and $K_0 = \mathbb{Q}(\sqrt{-d})$. One can find a finite extension K of K_0 , such that S has a model over K and that, moreover, the Néron-Severi group of S is generated by divisors which are defined over K . Shioda and Inose [SI77, Theorem 6] proved

Theorem 5.1. *Let S and K as above. Then*

$$L(S/K, s) \doteq \zeta_K(s-1)^{20} L(s, \psi^2) L(s, \overline{\psi}^2)$$

where $\zeta_K(s)$ is the Dedekind zeta function of the field K and ψ is the Grössencharacter of a model of the elliptic curve E_τ over K which is associated to S . Here \doteq denotes equality up to finitely many factors.

We shall now turn to the more special case that the singular K3 surface S is defined over \mathbb{Q} . Then the transcendental lattice $T(S)$ is a two-dimensional $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module, which defines an L -series $L(T(S), s)$. This case was treated by Livné [Liv95, Theorem 1.3].

Theorem 5.2. *Let S be a singular K3 surface of discriminant d , defined over \mathbb{Q} . Then there exists a weight three modular form f_3 with complex multiplication by $\mathbb{Q}(\sqrt{-d})$ such that*

$$L(T(S), s) \doteq L(f_3, s).$$

If, moreover, the Néron-Severi group $\text{NS}(S)$ is generated by divisors defined over \mathbb{Q} , then one has in fact that

$$L(S, s) \doteq \zeta(s-1)^{20} L(f_3, s).$$

The level of the form f_3 equals the discriminant of S up to squares composed of the bad primes [Schu05, Proposition 13.1].

An example where the latter situation occurs is the universal elliptic curve $\mathcal{E}_1(7)$ with a point of order 7. An affine model of this surface is given by

$$y^2 + (1+t-t^2)xy + (t^2-t^3)y = x^3 + (t^2-t^3)x^2.$$

In this case, $\text{NS}(\mathcal{E}_1(7))$ is generated by divisors defined over \mathbb{Q} , and the associated modular form is $f_3 = (\eta(\tau)\eta(7\tau))^3$, where η is the Dedekind η -function. For details of this example see [HV05b]. Another example of this kind was considered by Schütt and Top [ST05], who investigated the extremal elliptic fibration $\pi : S \rightarrow \mathbb{P}^1$ with singular fibres $I_{19}, I_1, I_1, I_1, I_1, I_1$.

At this point we would like to mention the following result, due to Shafarevich [Sha96].

Theorem 5.3. *Given an integer n , there is a finite set \mathcal{S}_n of $\text{SL}_2(\mathbb{Z})$ -equivalence classes of even, positive definite, binary quadratic forms, such that, if S is a singular K3 surface admitting a model over a number field of degree at most n then $T(S) \in \mathcal{S}_n$.*

In particular, this result says that there exist only finitely many \mathbb{C} -isomorphism classes of singular K3 surfaces which can be defined over \mathbb{Q} .

It is a natural question to ask which weight three newforms with rational coefficients can be realized by singular K3 surfaces defined over \mathbb{Q} . In view of Shafarevich's result, this number is finite (up to twisting). Motivated by this observation Schütt [Schu05] proved the following finiteness result for CM forms:

Theorem 5.4. *Assume the generalised Riemann hypothesis (GRH). Then, for given weight, there are (up to twisting) only finitely many CM newforms with rational coefficients. For weights 2, 3 and 4 this holds unconditionally.*

For weight two this is, of course, classical theory of CM elliptic curves. The fact that this theorem is true without the assumption of GRH for weights three and four follows from Weinberger's result [Wein73] of the finiteness of imaginary quadratic fields with exponent 2 or 3. Schütt has determined lists of such forms for weights three and four. For weight three this list is complete under the assumption of GRH and without this assumption the list is complete up to at most one form of level at least $2 \cdot 10^{11}$. For weight four the list is complete up to level 10^{10} . So far, he has found singular K3 surfaces over \mathbb{Q} for 24 out of the 65 weight three forms.

5.2. Rigid Calabi Yau threefolds. In this section we will introduce some of the rigid Calabi-Yau threefolds over \mathbb{Q} for which modularity has been proven. For a complete treatment, the reader is referred to the book of Meyer [Mey05]. Nevertheless, we tried to include the most important and instructive examples. Note that for all of them modularity statements can also be derived from Theorem 4.2 (although this theorem does not give the explicit modular form).

5.2.1. The Schoen quintic. To our knowledge the first rigid Calabi-Yau threefold over \mathbb{Q} for which modularity was proved is the Schoen quintic X in \mathbb{P}^4 :

$$X : x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - x_0 x_1 x_2 x_3 x_4 = 0.$$

Note that X appears as a (singular) member of the one-parameter family \mathcal{M} in Section 6.2. Schoen [Scho86] showed that X is rigid and derived the modularity of X explicitly:

Theorem 5.5. *X is modular and its L -series coincides with the Mellin transform of a normalised newform of weight four and level 25.*

Since the newform does not have even coefficients, the proof of the theorem requires Serre's approach (cf. Section 4.3). In this case, this is easy, since by [Jon] (cf. [Liv87, Proposition 4.10]) there is a unique Galois extension K/\mathbb{Q} with Galois group S_3 or C_3 which is unramified outside 2 and 5. By [Jon], there are only six extensions with Galois group S_4 or $S_3 \times C_2$ which have to be considered. One then finds that it suffices to check the primes in the set $T = \{3, 7, 11, 13\}$.

5.2.2. Fibre products of rational elliptic surfaces. A very useful method to construct Calabi-Yau threefolds goes back to Schoen. In [Scho88] he considers fibre products of two regular semi-stable rational elliptic surfaces with a section. Due to the semi-stability, such a fibre product has only ordinary double points as singularities, and hence a small resolution exists in the analytic category. Such a resolution has all properties of a Calabi-Yau variety, but may not be projective.

Sometimes, it is possible to construct a small projective resolution by successively blowing up suitable subvarieties. In general, there are two major classes of fibre products where enough subvarieties are available:

Proposition 5.6 (Schoen [Scho88, Lemma 3.1]). *For $i = 1, 2$ let $\pi_i : Y_i \rightarrow \mathbb{P}^1$ be a semi-stable rational elliptic surface with a section.*

Let X be the fibre product $Y_1 \times_{\mathbb{P}^1} Y_2$. Then X has a projective small resolution \hat{X} if

- (1) *all singularities of X lie in fibres of type $I_m \times I_n$ with $m, n > 1$, or*
- (2) *$Y_1 = Y_2$ and $\pi_1 = \pi_2$.*

Schoen also investigated when a resolution of a fibre product is rigid. This is independent of its projectivity. He describes four cases where this happens. In this section, we will see examples for two of them.

Let X be the self-fibre product of a semi-stable rational elliptic surface with the minimal number of four singular fibres. Then X has a projective small resolution \hat{X} by the proposition above. This is rigid due to [Scho88, Proposition 7.1, (i)]. There are six such surfaces up to isomorphism over \mathbb{Q} , as determined by Beauville [Be82]. All of them are modular in the sense of Shioda [Shi72], i.e., they are elliptic surfaces associated with a subgroup of $\mathrm{SL}(2, \mathbb{Z})$ of finite index. We give the corresponding

Γ	singular fibres	weight 4 form	level
$\Gamma(3)$	I_3, I_3, I_3, I_3	$\eta(3\tau)^8$	9
$\Gamma_1(4) \cap \Gamma(2)$	I_4, I_4, I_2, I_2	$\eta(2\tau)^4 \eta(4\tau)^4$	8
$\Gamma_1(5)$	I_5, I_5, I_1, I_1	$\eta(\tau)^4 \eta(5\tau)^4$	5
$\Gamma_1(6)$	I_6, I_3, I_2, I_1	$\eta(\tau)^2 \eta(2\tau)^2 \eta(3\tau)^2 \eta(6\tau)^2$	6
$\Gamma_0(8) \cap \Gamma_1(4)$	I_8, I_2, I_1, I_1	$\eta(4\tau)^{16} \eta(2\tau)^{-4} \eta(8\tau)^{-4}$	16
$\Gamma_0(9) \cap \Gamma_1(3)$	I_9, I_1, I_1, I_1	$\eta(3\tau)^8$	9

TABLE 1. Beauville surfaces

congruence subgroups in Table 1, where we also include the types of the singular fibres.

The fact that the elliptic surfaces can be defined over \mathbb{Q} , and that X is a self-fibre product imply that the small resolution \hat{X} of X is defined over \mathbb{Q} . Hence, we expect it to be modular.

There are three alternative proofs of modularity: one uses the Eichler-Shimura isomorphism and the fact that the respective spaces of newforms are one-dimensional. This has been worked out in [SY01]. The next approach consists of counting points and applying the Faltings-Serre-Livné method as sketched in Section 4.3. This was done by Verrill in [Yui03, Appendix]. We note that due to the self-fibre product structure, the number of points on the reductions and, consequently, the traces are always even, so Theorem 4.3 applies. Then one can use Theorem 4.2 and then try to proceed as in [Di04a].

Finally, take for every $\overline{\mathbb{Q}}$ -isomorphism class the elliptic surface given by the equations listed in [Be82]. Table 1 gives the weight four newforms associated to the self-fibre products and their respective levels.

We also remark that these newforms are the only ones of weight four which can be written as η -products (see [Mar96, Cor. 2]).

We shall give one further class of examples which was studied in [Schu04a]. As above, we shall work with the universal elliptic curve with a point of order 6, which we will denote by $\mathcal{E}_1(6)$. (This is the elliptic surface associated to $\Gamma_1(6)$.) We take $\pi_2 = \sigma\pi_1$ with σ an automorphism of \mathbb{P}^1 permuting the three cusps with reducible fibres and fixing the remaining cusp.

This leads to case 1 of Proposition 5.6. The projective small resolution \hat{X} is rigid by [Scho88, Proposition 7.1, (ii)]. There are five such automorphisms of \mathbb{P}^1 which are all defined over \mathbb{Q} . The resulting rigid Calabi-Yau threefolds can be proved to be modular and one determines the corresponding newforms using Theorem 4.3. The associated newforms are the forms of level 10, 17, 21, and twice the same form of level 73.

5.2.3. Further examples. We shall briefly comment on other modular rigid Calabi-Yau threefolds. To our knowledge, the first examples which were published after the Schoen quintic were obtained by Werner and Van Geemen [WvG90], based on work of Hirzebruch. Their paper pioneered an important method to determine the third Betti number $b_3(X)$ for a Calabi-Yau threefold X over \mathbb{Q} arithmetically. This method requires the knowledge of the Euler number $e(X)$ and of a good prime p such that Fr_p operates trivially on the divisors of X . Then the Riemann hypothesis gives a bound in terms of $b_3(X)$ on the trace of Fr_p on $H_{\text{ét}}^3(X, \mathbb{Q}_l)$. This bound

sometimes enables one to deduce the value of $b_3(X)$ after counting points. Using this approach and Theorem 4.3, Werner and van Geemen proved modularity for four rigid Calabi-Yau threefolds over \mathbb{Q} . They also determined the associated newforms, finding the forms of level 8 and 9 from above and two further forms of level 12 and 50 respectively.

Further, we would like to mention two constructions which produced several modular Calabi-Yau threefolds. The first of them uses toroidal geometry involving root lattices. For the root lattices A_1^3 and A_3 , this was worked out by Verrill in [Ver00]. The corresponding newforms of level 6 and 8 appear in Table 1. For A_4 , a family of Calabi-Yau threefolds was investigated by Hulek and Verrill in [HV03]. This family contains a number of (rigid and non-rigid) members for which they prove modularity.

Another approach uses double octics, as studied by Cynk and Meyer in [CM05a], [CM05c]. In this case one starts with an octic surface S in \mathbb{P}^3 which has mild singularities. Many such double covers of \mathbb{P}^3 admit a desingularisation X , which is Calabi-Yau. In practice one uses for S highly reducible surfaces, such as a union of 8 planes (such configurations are also called *octic arrangements*). Such an X is called a double octic. Often it is possible to describe the deformations of a double octic X explicitly in terms of the surface S . Using this, Cynk and Meyer found further rigid Calabi-Yau threefolds over \mathbb{Q} and the corresponding newforms in [CM05a].

One advantage of the constructions involving fibre products, root lattices or double octics is that they also produce non-rigid modular Calabi-Yau threefolds. Here, modularity is understood as the splitting of $H_{\text{ét}}^3(X, \mathbb{Q}_l)$ into two-dimensional Galois representations, all of which can be interpreted in terms of modular forms. In practice, this is often achieved by identifying enough elliptic ruled surfaces in X which contribute to $H_{\text{ét}}^3(X, \mathbb{Q}_l)$. This method is explained in [HV05a]. For applications, the reader is also referred to [HV03], [Schu04b] and [CM05c].

Apart from this strict notion of modularity, the Langlands' programme hardly seems to be within reach in more generality at the moment. In [CS01], Consani-Scholten investigate a Calabi-Yau threefold X over \mathbb{Q} with $b_3(X) = 4$. They prove that the Galois representation on $H_{\text{ét}}^3(X, \mathbb{Q}_l)$ is induced from a two-dimensional representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5}))$ and conjecture that this should come from a Hilbert modular form over $\mathbb{Q}(\sqrt{5})$ of weight $(2, 4)$ and level 5. Recently Yi [Yi05] has announced a proof of this conjecture.

5.3. Higher-dimensional examples. A number of examples of higher-dimensional modular Calabi-Yau manifolds were recently constructed in [CH05]. Most of these examples arise as quotients of products of lower dimensional Calabi-Yau manifolds by finite groups. The starting point is the Kummer construction, which was first used by Borcea [Bor92] and Voisin [Voi93], who constructed Calabi-Yau threefolds as (birational) quotients of a product of a K3 surface admitting an elliptic fibration and an elliptic curve, by the diagonal involution. This construction works in all dimensions: let Y be a projective manifold with $H^q(Y, \mathcal{O}_Y) = 0$ for $q > 0$ and $D \in |-2K_Y|$ be a smooth divisor. Then D defines a double cover $\pi : X \rightarrow Y$, branched along D with X a Calabi-Yau manifold. Now assume that we have two coverings $\pi_i : X_i \rightarrow Y_i$ as above. Then the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on the product $X_1 \times X_2$.

Proposition 5.7. *Under the above assumptions, the quotient of the product $X_1 \times X_2$ by the diagonal involution admits a crepant resolution X , which is Calabi-Yau. Moreover, there is a double cover $X \rightarrow Y$, branched along a smooth divisor D , with $H^q(Y, \mathcal{O}_Y) = 0$ for $q > 0$.*

The last remark of this proposition allows one to use the covering $\pi : X \rightarrow Y$ to repeat this process inductively. Starting with low-dimensional modular Calabi-Yau varieties, one can in this way easily construct many higher dimensional Calabi-Yau varieties. The middle cohomology of these varieties grows with the dimension, but splits in many cases into two-dimensional pieces, which allows to prove modularity. Here we give one example. As the first factor we choose the rigid Calabi-Yau threefold X_3 , constructed as a resolution of singularities of the double covering of \mathbb{P}^3 , branched along the following arrangement of eight planes

$$xt(x-z-t)(x-z+t)y(y+z-t)(y+z+t)(y+2z) = 0.$$

For details of this example see [Mey05, Octic Arr. No. 19]. As the second factor we take the K3 surface S which is obtained as a desingularisation of the double sextic branched along the following arrangement of six lines

$$xy(x+y+z)(x+y-z)(x-y+z)(x-y-z) = 0.$$

Performing the Kummer construction just described, we obtain a smooth Calabi-Yau fivefold X_5 . The Hodge groups of X_5 are the invariant part of the Hodge groups of $X_3 \times S$. From this it follows easily that $b_1(X_5) = b_3(X_5) = 0$ and $b_5(X_5) = 4$. More precisely

$$H^5(X_5) \cong H^3(X_3) \otimes T(S)$$

where $T(S)$ is the transcendental lattice of S . It is known that

$$L(T(S), s) \doteq L(g_3, s), \quad L(X_3, s) \doteq L(g_4, s)$$

where g_3 and g_4 are the unique weight three and weight four Hecke eigenforms of level 16 and 32 respectively, with complex multiplication.

Both of these forms can be derived from the unique weight two level 32 newform

$$g_2(q) = \eta(q^8)^2 \eta(q^4)^2$$

by taking the second, respectively, third power of the corresponding Grössencharacter ψ of conductor $2+2i$. The $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module $H_{\text{ét}}^5(\overline{X}, \mathbb{Q}_l)$ is the tensor product of the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module $H_{\text{ét}}^3(\overline{X}, \mathbb{Q}_l)$ and of the transcendental lattice $T(S)$ of S . Its semi-simplification splits into two two-dimensional pieces and one finds for the L -series

$$L(X_5, s) \doteq L(g_4 \otimes g_3, s) = L(g_6, s)L(g_2, s-2)$$

where g_2 is as above, and g_6 is a level 32 cusp form of weight 6, which can be derived from g_2 by taking the fifth power of the Grössencharacter ψ .

If one wants to find examples of higher dimensional Calabi-Yau manifolds which are the analogue of rigid Calabi-Yau varieties, i.e., have two-dimensional middle cohomology, then one has to divide by more complicated groups. One example of such a construction is the following. Consider the elliptic curve

$$E : y^2 = x^3 - D$$

for some positive integer D . Then $x \mapsto \rho x$, with ρ a third root of unity, defines an automorphism η of order 3 on E .

Theorem 5.8. *Let $n > 1$ be an integer and let X' be the quotient of the cartesian product E^n by the group*

$$\{(\eta^{a_1} \times \cdots \times \eta^{a_n}) \in \text{Aut}(E^n); a_1 + \cdots + a_n \equiv 0 \pmod{3}\}.$$

Then X' has a smooth model X which is a Calabi-Yau manifold and the dimension of $H_{\text{ét}}^n(\overline{X}, \mathbb{Q}_\ell)$ equals two if n is odd, respectively the dimension of $T(X) \otimes \mathbb{Q}_\ell$ equals two if n is even, where $T(X)$ is the transcendental part of the middle cohomology.

Moreover, X is defined over \mathbb{Q} and $L(H_{\text{ét}}^n(\overline{X}, \mathbb{Q}_\ell), s) \doteq L(g_{n+1}, s)$, respectively $L(T(X) \otimes \mathbb{Q}_\ell, s) \doteq L(g_{n+1}, s)$, where g_{n+1} is the weight $n+1$ cusp form with complex multiplication in $\mathbb{Q}(\sqrt{-3})$, associated to the n -th power of the Grössencharacter of E .

One can also attempt to generalise the construction of double covers of \mathbb{P}^n , branched along a configuration of hyperplanes, to higher dimension. In general, however, this becomes very difficult. The combinatorial problems in order to control the singularities of, say, a double cover of \mathbb{P}^5 branched along 12 hyperplanes become very hard. In [CH05] one example was studied, namely the double cover X of \mathbb{P}^5 given by

$$w^2 = x(x-u)(x-v)y(y-u)(y-v)z(z-u)(z-v)t(t-u)(t-v).$$

This example was first studied by Ahlgren, who counted the number of points modulo p on the affine part of X , relating these numbers to a modular form. The following was proved in [CH05, Theorem 5.12].

Theorem 5.9. *The variety X has a smooth model Z , defined over \mathbb{Q} , which is a Calabi-Yau fivefold with Betti numbers $b_1(Z) = b_3(Z) = 0$, $b_5(Z) = 2$. More precisely, $h^{5,0} = h^{0,5} = 1$. One has*

$$L(Z, s) \doteq L(f_6, s)$$

where $f_6(q) = \eta^{12}(q^2)$ is the unique normalised cusp form of level four and weight six.

6. ZETA FUNCTIONS AND MIRROR SYMMETRY

6.1. p -adic cohomology. In practice, it turns out to be hard to work with étale cohomology in an explicit way (at least in dimension two and higher). In this subsection we discuss another approach to calculate the zeta-function of a smooth variety over a finite field \mathbb{F}_q . This gives enough information to obtain the characteristic polynomial of Fr_p on $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$ for a prime of good reduction. We use cohomology with p -adic coefficients instead of l -adic coefficients. Cohomology theories with p -adic coefficients occurred, for example, in Dwork's proof [Dwo60] of the rationality of the zeta-function.

There exist several p -adic cohomology theories which are rich enough to prove the Weil conjectures, and therefore are called 'Weil cohomologies'. These theories have a Lefschetz trace formula.

We discuss an easy variant, called Monsky-Washnitzer cohomology, which strictly speaking is not a Weil cohomology. For a survey on this theory we refer to [vdP86].

Let p be a prime number, r a positive integer. Set $q = p^r$. Suppose V/\mathbb{F}_q is a hypersurface in \mathbb{P}^n and let X be the complement of V in \mathbb{P}^n . Then X is an affine variety, say $X = \text{Spec } R'$. Consider the ring \mathbb{Z}_q , the ring of integers of \mathbb{Q}_q , which is the unique unramified extension of \mathbb{Q}_p of degree r . Let π be the

maximal ideal of \mathbb{Z}_q . The ring R' can be ‘lifted’ to \mathbb{Z}_q , i.e., there is an algebra $\hat{R} := \mathbb{Z}_q[X_0, \dots, X_m]/(f_1, \dots, f_k)$ such that $\hat{R}/\pi\hat{R} \cong R'$.

Let R be the ‘overconvergent completion’ of \hat{R} , that is the ring

$$\frac{\{g \in \mathbb{Z}_q[[X_0, \dots, X_m]]; g \text{ converges on an open disc of radius more than } 1\}}{(f_1, \dots, f_k)}.$$

This definition enables us to have a good theory of differential forms on X : consider the complex of differential algebras

$$0 \rightarrow \Omega_R^0 \rightarrow \Omega_R^1 \rightarrow \Omega_R^2 \rightarrow \dots \rightarrow \Omega_R^n \rightarrow 0.$$

Then we define $H_{MW}^i(X, \mathbb{Q}_q)$ to be the i -th cohomology group of this complex tensored with \mathbb{Q}_q . It turns out that these groups are finite-dimensional. The reason to work with (overconvergent) p -adic algebras is that they admit lifts of Frobenius, i.e., there exists a morphism $\text{Fr}_q : R \rightarrow R$ such that the reduction $\text{Fr}_q : R/\pi R \rightarrow R/\pi R$ is the geometric Frobenius map on X .

One can show that

$$\sum_i (-1)^i \text{tr}(q^n \text{Fr}_q^{-1} | H_{MW}^i(X, \mathbb{Q}_q)) = \#X(\mathbb{F}_q).$$

The appearance of $q^n \text{Fr}_q^{-1}$ comes from the fact that Monsky-Washnitzer cohomology is the Poincaré dual of rigid cohomology with compact support. The latter theory has a more usual Lefschetz trace formula. We note that R and Fr_q are far from being unique. Let $(R_1, \text{Fr}_{q,1})$ be another lift. Then there is a canonical isomorphism ψ between the cohomology groups associated to Ω_R^\bullet and the cohomology groups associated to $\Omega_{R_1}^\bullet$ such that $\psi \circ \text{Fr}_q = \text{Fr}_{q,1} \circ \psi$.

Example 6.1. Let V be the curve given by $F = y^2z - x^3 - axz^2 - bz^3$ for $a, b \in \mathbb{F}_q$ such that $\gcd(6, q) = 1$ and $4a^3 + 27b^2 \neq 0$. Let $X = \mathbb{P}^2 \setminus V$. Then $H_{MW}^0(X, \mathbb{Q}_q) \cong \mathbb{Q}_q$ and $H_{MW}^2(X, \mathbb{Q}_q)$ is generated by

$$\frac{xyz}{F}\Omega \text{ and } \frac{(xyz)^2}{F^2}\Omega$$

with $\Omega = dy \wedge dz - dx \wedge dz + dx \wedge dy$. All other cohomology groups vanish.

As a lift of Frobenius one can take $[x, y, z] \mapsto [x^p, y^p, z^p]$.

Remark 6.2. Suppose $E \subset \mathbb{P}_{\mathbb{C}}^2$ is a complex elliptic curve given by the zero-set of a cubic F as in the above example. Then one can show that the usual de Rham cohomology group $H^2(\mathbb{P}^2 \setminus E, \mathbb{C})$ is generated by

$$\omega_1 := \frac{XYZ}{F}\Omega \text{ and } \omega_2 := \frac{(XYZ)^2}{F^2}\Omega.$$

A famous theorem of Griffiths [Gri69] applied to this case yields that ω_2 generates $F^2 H^2(\mathbb{P}^2 \setminus E, \mathbb{C})$ and $F^1 H^2(\mathbb{P}^2 \setminus E, \mathbb{C})/F^2 H^2(\mathbb{P}^2 \setminus E, \mathbb{C})$ is generated by ω_1 , where F^\bullet is the Hodge filtration.

Example 6.3. Let g be a positive integer. Consider the algebra

$$\bar{R} := \mathbb{F}_q[x, y, z]/(-y^2 + f_{2g+1}, yz - 1)$$

with $f_{2g+1} \in \mathbb{F}_q[x]$ a polynomial of degree $2g+1$, with only simple roots. Then $\text{Spec } \bar{R}$ is a hyperelliptic curve of genus g minus the $2g+2$ Weierstrass points. Lift $\bar{0}, \bar{1}, \dots, \bar{p-1}$ to $0, 1, \dots, p-1$. This choice induces a unique lift $\sigma : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ of Frobenius.

A basis for $H_{MW}^1(X, \mathbb{Q}_q)$ is

$$x^i dx \text{ for } i = 0, \dots, 2g, \text{ and } x^i \frac{dx}{y} \text{ for } i = 0, \dots, 2g - 1.$$

There is a unique lift Fr_q of Frobenius extending σ , such that $\text{Fr}_q(x) = x^q$, $\text{Fr}_q(y) \equiv y^q \pmod{\pi}$ and $\text{Fr}_q(y)^2 = \text{Fr}_q(f_{2g+1})$. Finally $\text{Fr}_q(z)$ is defined by $\text{Fr}_q(z)\text{Fr}_q(y) = 1$. This description of the Monsky-Washnitzer cohomology enabled Kedlaya [Ked01] to present an efficient point counting algorithm on hyperelliptic curves over fields of small characteristic. We would like to point out that $\text{Fr}_q(y)$ and $\text{Fr}_q(z)$ are overconvergent power series which do not converge to a rational function. This is in contrast to the previous example.

The fact that we are working with a cohomology theory using differential forms enables us to use more analytic techniques. Consider the family V_t of elliptic curves $X^3 + Y^3 + Z^3 + 3tXYZ$ where t is a parameter in the p -adic unit disc Δ . Then $H_{MW}^2(X_{t_0}, \mathbb{Q}_q)$ is two-dimensional for a general $t_0 \in \Delta$. One of the two eigenvalues of $p^2 \text{Fr}_p^{-1}$ has p -adic valuation 0. This eigenvalue is called the unit root. Dwork showed that the unit root, as a function of t , satisfies the p -adic hypergeometric differential equation with parameters $\frac{1}{2}, \frac{1}{2}; 1$. Specialising to values t_0 on the boundary of the unit disc gives the characteristic polynomial of Frobenius on $H_{MW}^2(X_{\overline{t_0}}, \mathbb{Q}_q)$ with $\overline{t_0} \equiv t \pmod{\pi}$. N. Katz has generalised this idea: on the p -adic cohomology groups $H_{MW}^{n+1}(X_t)$ one can introduce a Gauss-Manin connection, much in the same way as one usually does in the complex case.

Theorem 6.4 (Katz, [Kat68]). *Let X_t be a family of hypersurface complements in \mathbb{P}^n and let $(H_{MW}^n(X_t, \mathbb{Q}_q), \nabla_{GM})$ be the local system of cohomology groups. Let $A(t)$ be a solution of the Picard-Fuchs equation (i.e., the differential equation associated to ∇_{GM} .) Fix a basis $e_{1,t}, \dots, e_{m,t}$ for this local system and let $\text{Fr}(t)$ be the matrix of the Frobenius action on $H_{MW}^n(X_t, \mathbb{Q}_q)$ with respect to $e_{1,t}, \dots, e_{m,t}$. Let $t_0 \in \mathbb{Q}_q$ such that $|t_0|_q = 1$. Then*

$$\text{Fr}(t_0) = A(t_0^q)^{-1} \text{Fr}(0) A(t_0)$$

provided that the reduction $\overline{X}_{\overline{t_0}}$ is smooth.

6.2. Mirror Symmetry and Zeta Functions. The classical Picard-Fuchs equations of Calabi-Yau threefolds play an important role in Mirror Symmetry. Candelas, de la Ossa and Rodriguez-Villegas [COR00], [COR03] considered the p -adic Picard-Fuchs equation (cf. Theorem 6.4) of the following one-parameter family of quintics

$$(5) \quad \mathcal{M}: \quad Q(\psi) = \left\{ \sum_{i=0}^4 x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \right\}.$$

For general ψ , i.e., $\psi^5 \neq 0, 1, \infty$, this is a smooth quintic. They calculated explicit expressions for the number of points on $Q(\psi)$, using tedious computations with so-called Dwork characters. Then they observed that these expressions were solutions to the Picard-Fuchs equations. The above mentioned Theorem 6.4 gives a more mathematical explanation for this phenomenon.

Using these calculations, Candelas et al. determined the structure of the zeta-function, more precisely, they showed that for any prime $p \neq 5$ and any $\psi \in$

$$\mathbb{F}_p \setminus \{0, 1\}$$

$$(6) \quad Z_{\mathcal{M}}(t, \psi) = Z_{Q(\psi)}(t) = \frac{R_0(t, \psi) R_A(p^\rho t^\rho, \psi)^{\frac{20}{\rho}} R_B(p^\rho t^\rho, \psi)^{\frac{30}{\rho}}}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}$$

where $\rho = 1, 2$ or 4 is the least integer such that $p^\rho \equiv 1 \pmod{5}$. The R 's are quartic polynomials in t .

We would like to point out that this factorisation can be obtained in a more direct way. As far as we know this strategy is not contained in the literature. Every member of the family \mathcal{M} has many automorphisms. We distinguish three types. First of all, each $\sigma \in S_5$ induces a permutation of coordinates $x_i \mapsto x_{\sigma(i)}$. Secondly, let ζ be a primitive 5-th root of unity. Then the group

$$(7) \quad G = \left\{ \text{diag}(\zeta^{n_0}, \zeta^{n_1}, \zeta^{n_2}, \zeta^{n_3}, \zeta^{n_4}); \sum n_i \equiv 0 \pmod{5} \right\}$$

acts on the family \mathcal{M} . (Note that this action is defined over \mathbb{F}_{p^ρ} .) Finally, the third type is the Frobenius automorphism Fr_{p^ρ} . Note that Fr_{p^ρ} commutes with all elements in G as well as all elements in S_5 . It is easy to write down an explicit basis for $H_{MW}^4(\mathbb{P}^4 \setminus Q(\psi), \mathbb{Q}_p)$ and to calculate the action of the automorphisms of the first and second type on this basis.

Since two commuting automorphisms respect each other's eigenspaces, we can use the automorphisms coming from the S_5 -action to identify all the eigenspaces of Frobenius on $H_{MW}^4(\mathbb{P}^4 \setminus Q(\psi), \mathbb{Q}_p)$. Then we can use the automorphisms of G to prove that many of the eigenvalues of Frobenius coincide. This gives a factorisation of $Z_{\mathbb{P}^4 \setminus Q(\psi)}(t)$. Since $Z_{\mathbb{P}^4 \setminus Q(\psi)}(t) Z_{\mathcal{M}}(t, \psi) = Z_{\mathbb{P}^4}(t)$ this factorisation can be translated in a factorisation of $Z_{\mathcal{M}}(t, \psi)$ as in (6).

The R 's are quartic polynomials, satisfying the functional equation

$$R\left(\frac{1}{p^3 t}, \psi\right) = \frac{1}{p^6 t^4} R(t, \psi).$$

In particular

$$R_0(t, \psi) = 1 + a_1(\psi)t + b_1(\psi)pt^2 + a_1(\psi)p^3t^3 + p^6t^4$$

where $a_1(\psi), b_1(\psi)$ depend on the parameter ψ . The two factors R_A and R_B are closely related to genus 4 curves $A = A(\psi)$ and $B = B(\psi)$, namely

$$Z_A(u, \psi) = \frac{R_A(u, \psi)^2}{(1-u)(1-pu)},$$

with an analogous expression for R_B . The curves A and B are the nonsingular models of

$$y^5 = x^2(1-x)^\beta(1-x/\psi^5)^{5-\beta}$$

where $\beta = 3$ for A , and $\beta = 4$ for B . The geometric role of A and B does not seem to be well understood.

There is no good notion of a mirror family over non-algebraically closed fields in positive characteristic. However, in this case we can mimic the mirror construction since the mirror family of (5), considered as a family of complex varieties, is given by a quotient construction using G : for a general element $Q(\psi)$ the quotient $Q(\psi)/G$ has nodes and resolving these nodes gives Calabi-Yau threefolds $Q'(\psi)$, which form the mirror family \mathcal{W} of \mathcal{M} .

In the complex case one has an equivalent construction, namely as family of hypersurfaces in the toric variety \mathbb{P}_Δ where Δ is the simplex in \mathbb{R}^4 given by

$\{e_1, \dots, e_4, -(e_1 + \dots + e_4)\}$. The mirror family is then given by crepant resolutions of the closure in \mathbb{P}_Δ of the affine family in the torus $(\mathbb{C}^*)^4$ defined by the equation

$$x_1 + \dots + x_n + \frac{1}{x_1 \dots x_4} + \psi = 0.$$

It is known among experts how one can generalise this construction to p -adic toric varieties.

Candelas et al. computed the zeta function of the mirror family for general ψ as

$$(8) \quad Z_{\mathcal{W}}(t, \psi) = Z_{Q'(\psi)}(t) = \frac{R_0(t, \psi)}{(1-t)(1-pt)^{101}(1-p^2t)^{101}(1-p^3t)}.$$

Using the explicit differential forms provided by Monsky-Washnitzer cohomology one can easily find an explicit basis for $H_{MW}^4(\mathbb{P}^4 \setminus Q(\psi))^G$. Along the same lines one shows that $R_0(t, \psi)$ is the characteristic polynomial of Frobenius on $H_{MW}^4(\mathbb{P}^4 \setminus Q(\psi))^G$. This explains why $R_0(t, \psi)$ is a factor of the numerator of the Zeta function of the mirror threefold.

Comparing equations (6) and (8) shows that there is a close relationship between the zeta function of the family \mathcal{M} and its mirror \mathcal{W} . In particular, one obtains the congruence

$$(9) \quad Z_{\mathcal{M}}(t, \psi) \equiv \frac{1}{Z_{\mathcal{W}}(t, \psi)} \equiv (1-pt)^{100}(1-p^2t)^{100} \pmod{5^2}.$$

In her thesis Kadir [Kad04] has investigated the case of double octics, where she was able to prove analogous results.

The question of studying the zeta functions of mirror families was also taken up by Wan [Wan04] and Fu and Wan [FW05]. Wan considered the family of hypersurfaces

$$X_\psi : x_0^{n+1} + \dots + x_n^{n+1} + \psi x_0 \dots x_n = 0.$$

The construction of the ‘mirror family’ Y_ψ is then completely analogous to the quintic case. He calls a pair (X_ψ, Y_ψ) with the same parameter ψ a *strong* mirror pair and proves the following generalisation of (9) for such pairs:

$$(10) \quad \#X_\psi(\mathbb{F}_{q^k}) \equiv \#Y_\psi(\mathbb{F}_{q^k}) \pmod{q^k}.$$

Here $q = p^r$ is a prime power. He conjectures that such congruences should hold in greater generality. Such a generalisation for certain pairs $(X, X/G)$ was then indeed obtained by Fu and Wan in [FW05]. As a consequence of their results they prove

Theorem 6.5. *Let X be a smooth, geometrically connected variety X over the finite field \mathbb{F}_q , $q = p^r$. Assume that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Then for all integers k , the congruence*

$$\#X(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k}$$

holds. Moreover, let G be a finite group of \mathbb{F}_q automorphisms acting on X . Then

$$\#(X/G)(\mathbb{F}_{q^k}) \equiv \#X(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k}.$$

Their proof uses crystalline cohomology and the Mazur-Ogus theorem. Esnault [Esn05] has pointed out that these results can also be obtained using de Rham-Witt cohomology and rigid cohomology.

7. OPEN QUESTIONS

We conclude this paper with a brief discussion of some open problems. These will mainly concern Calabi-Yau threefolds. We start with some questions related to modular forms.

If f is a newform of weight two with field of coefficients F , then a classical construction associates to f an abelian variety A over \mathbb{Q} of dimension $[F : \mathbb{Q}]$. In particular, if f has rational coefficients, then this gives an elliptic curve E . For higher weight, however, no such construction is available. As remarked in Remark 3.1 one can always find some symmetric product of some universal curve for which part of the cohomology corresponds to f . One might ask whether any such form can be realised geometrically by a Calabi-Yau variety:

Question 7.1. Let f be a newform of weight $k > 2$ with rational Fourier coefficients. Is there a Calabi-Yau variety X over \mathbb{Q} such that $L(f, s)$ occurs in the L -series of X ?

This question was formulated independently by B. Mazur and D. van Straten.

Consider two non-isomorphic modular varieties defined over \mathbb{Q} such that they are isomorphic over $\overline{\mathbb{Q}}$. In practice, it turns out that very often the associated modular forms differ by a twist. E.g., this can be seen in the case of elliptic fibrations in Weierstrass form or with double octics, where twists over quadratic extensions can easily be realised. For this reason the above question should be asked modulo twisting. By Theorem 5.4 (and assuming GRH) the number of available forms of given odd weight is finite up to twisting. In the case of K3 surfaces Shafarevich's theorem 5.3 also tells us that the number of available surfaces is finite, hence in the weight three case this reduces to a question of matching two finite sets. We have already mentioned that, so far, 24 of the 64 forms have been realised geometrically (see Subsection 5.1 or [Schu05, Section 15]).

On the other hand, if the weight k is even, then there are infinitely many newforms with rational coefficients, even up to twisting. An answer to Question 7.1 would therefore require a general construction, similar to the weight two case of elliptic curves. In the case of weight four, we have seen a number of examples of modular Calabi-Yau threefolds in Section 5.2. In total, Calabi-Yau threefolds for around 80 newforms have been found so far (modulo twisting). We refer the reader to Meyer's book [CM05a], which contains the most systematic collection of examples up to date. These include non-rigid and nodal varieties as well as some threefolds where the corresponding newform is only conjectural. Nevertheless, there are no known Calabi-Yau partners for the newforms of level 7 and 13. In general, it seems to be hard to find modular varieties with big bad primes. It would be interesting to find constructions which yield such examples.

Another question concerns the level of the corresponding newform:

Question 7.2. Let X be a modular variety over \mathbb{Q} . Can one predict the level of the associated newform?

In particular, this question concerns the bad primes of X . Once more, the difference with the special case of elliptic curves has to be emphasised. For an elliptic curve E , say over \mathbb{Q} and hence modular, the theory provides a globally minimal model. This can be used to define the conductor of the curve, which then coincides with the level of the corresponding newform.

For a general variety X over \mathbb{Q} , no concept of arithmetic minimality is known. Nevertheless, we can define a conductor for any compatible system of étale Galois representations. Assuming modularity, this conductor coincides with the level N of the associated newform. This is a consequence of the compatibility with the local Langlands correspondence.

In Section 4.1, we gave the classical bounds for the exponents e_p of the conductor, if E is an elliptic curve. The same bounds also hold in many other cases. For a smooth projective variety X over \mathbb{Q} of odd dimension n , these were proved by Serre in [Ser87] for the Galois representation of $H_{\text{ét}}^n(X, \mathbb{Q}_l)$ provided this is two-dimensional and the Hodge decomposition is

$$H^n(X_{\mathbb{C}}, \mathbb{C}) \cong H^{n,0}(X_{\mathbb{C}}) \oplus H^{0,n}(X_{\mathbb{C}}),$$

and assuming that the representation is modular. This was reproved by Dieulefait in [Di04a]. Dieulefait also remarks that his proof (combined with the potential modularity result of Taylor) is enough to conclude that the result still holds without the modularity assumption. In particular, these bounds apply to rigid Calabi-Yau threefolds over \mathbb{Q} . For newforms of odd weight and rational coefficients (or in general CM-forms), the same bounds have been obtained by Schütt in [Schu05, Cor. 12.2]. In practice, these bounds can be used to precisely determine the corresponding newform, if modularity is known (e.g., by Theorem 4.2). One then writes down all newforms with level composed of the bad primes and checks some coefficients after counting points. For more details we refer to [Di04a], where this was discussed for Calabi-Yau varieties. At the moment, this method only works if there are at most two very small bad primes or if the newform has CM.

This takes us back to singular K3 surfaces and rigid Calabi-Yau threefolds over \mathbb{Q} . If a bad prime p divides the level of the newform associated to X , then examples show the following: if the reduction X_p has a single A_1 -singularity then $e_p = 1$. Otherwise, $e_p = 2$ (for $p > 3$) or $e_p \geq 2$, if $p = 2$ or 3). In view of this, we would like to ask the following question:

Question 7.3. Let X be a modular variety over \mathbb{Q} with corresponding newform f . Under which conditions do the bad Euler factors of $L(X, s)$ and $L(f, s)$ coincide?

Since the conductors have to be equal, one can hope for an unconditional answer. To our knowledge, this has not yet been investigated.

We shall now mention one further tool which can sometimes be used to prove modularity. This consists in exhibiting a birational map (over \mathbb{Q}) between a modular variety and one which is expected to be modular. In [HSGS01], two such varieties were called *relatives*. This paper gives a number of examples for such relatives. For example, birational maps over \mathbb{Q} between the following varieties were given: the Barth-Nieto quintic, the self-fibre product of the universal elliptic curve $\mathcal{E}_1(6)$ and Verrill's threefold associated to the root lattice A_3 . Recently, Cynk and Meyer published a study of rigid Calabi-Yau threefolds of level 8 where the question of relatives was treated systematically [CM05b].

This issue can also be discussed for the self-fibre products from Section 5.2.2. Here, each permutation of four cusps leads to isogenous elliptic fibrations. Then it is a matter of the field of definition of the isogeny which newform occurs. If the isogeny is not defined over \mathbb{Q} , then the newform might be twisted. For instance, the modular elliptic surfaces associated to $\Gamma_1(4) \cap \Gamma(2)$ and $\Gamma_0(8) \cap \Gamma_1(4)$ are isogenous.

If one works with the equations from [Be82], this isogeny is only defined over $\mathbb{Q}(i)$. The newforms of level 8 and 16 are twists by the corresponding quadratic character.

In general, this problem is addressed by the *Tate conjecture*. Let ρ_1, ρ_2 be isomorphic two-dimensional l -adic Galois representations coming from geometry. In particular, these Galois representations occur in the étale cohomology of some smooth projective varieties X_1, X_2 over \mathbb{Q} . Then the Tate conjecture predicts a correspondence, i.e., an algebraic cycle in the product $X_1 \times X_2$, which is defined over \mathbb{Q} and which induces an isomorphism between ρ_1 and ρ_2 .

For details on known correspondences, the reader is referred to the book of Meyer [Mey05]. Here we shall only mention one particular example. In [vGN95], Van Geemen and Nygaard study three Siegel modular threefolds over \mathbb{Q} . One of them is the first non-rigid Calabi-Yau threefold for which modularity has been established. Denoting this variety by X , Van Geemen and Nygaard show that $h^{1,2}(X) = 1$. An automorphism of X is used to split the four-dimensional Galois representation of $H_{\text{ét}}^3(X, \mathbb{Q}_l)$ into a sum of two-dimensional representations. Then the authors deduce the equality

$$L(X, s) = L(f_4, s) L(f_2, s - 1)$$

with newforms of level 32 and weight four and two respectively. Here, f_2 comes from the Grössencharacter ψ of $\mathbb{Q}(i)$ with conductor $(2 + 2i)$ and ∞ -type 1. The other newform, f_4 is derived from ψ^3 . On the other hand, ψ is related to the elliptic curve with CM by $\mathbb{Z}[i]$:

$$E: y^2 = x^3 - x \quad \text{as} \quad L(E, s) = L(\psi, s).$$

As a consequence,

$$L(X, s) = L(\text{Sym}^3 H_{\text{ét}}^1(E, \mathbb{Q}_l), s),$$

and hence the Tate conjecture predicts the existence of a correspondence between X and $E \times E \times E$ which induces an isomorphism between the two representations corresponding to ψ^3 . This is established by means of a curve of genus 5 over \mathbb{Q} which admits rational maps over \mathbb{Q} to both varieties, X and $E \times E \times E$.

However, there are modular varieties X and Y with the same associated form, for which there is no known correspondence.

Question 7.4. Let X and Y be modular varieties which have an associated form in common, find a correspondence between them.

We would like to conclude this section with a question related to arithmetic mirror symmetry. In Section 6.2 we discussed the zeta function of the family \mathcal{M} of quintics in \mathbb{P}^4 (See (6)). This expression involves genus 4 curves A and B whose geometric meaning has, so far, remained a mystery.

Question 7.5. What is the geometric meaning of the genus 4 curves which appear in the motive of the one-parameter quintic family \mathcal{M} ?

REFERENCES

- [Bat94] V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Alg. Geom.* **3**(1994), 493–535.
- [BB96] V. V. Batyrev and L. A. Borisov, On Calabi-Yau complete intersections in toric varieties, In *Higher-Dimensional Complex Varieties (Trento, 1994)*, 39–65, de Gruyter, Berlin, 1996.
- [Be82] A. Beauville, Les familles stables de courbes elliptiques sur \mathbb{P}^1 admettant quatre fibres singulières, *C.R. Acad. Sci. Paris* **294** (1982), 657–660.

- [Bor92] C. Borcea, Calabi-Yau threefolds and complex multiplication, In *Essays on mirror manifolds*, 489–502, Internat. Press, Hong Kong, 1992.
- [BCDT01] C. Breuil, B. Conrad, F. Diamond, R. Taylor, On the modularity of elliptic curves over \mathbf{Q} : wild 3-adic exercises. *J. Amer. Math. Soc.* **14** (2001), 843–939.
- [COR00] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, Calabi-Yau manifolds over finite fields I, hep-th/0012233. 2000.
- [COR03] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, Calabi-Yau manifolds over finite fields II, In *Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001)*, 121–157, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.
- [CS01] C. Consani and J. Scholten, Arithmetic on a quintic threefold, *Internat. J. Math.* **12** (2001), 943–972.
- [CH05] S. Cynk and K. Hulek, Construction and examples of higher-dimensional modular Calabi-Yau manifolds, preprint, math.AG/050942. 2005.
- [CM05a] S. Cynk and C. Meyer, Geometry and arithmetic of certain double octic Calabi-Yau manifolds, *Canad. Math. Bull.* **48** (2005), 180–194.
- [CM05b] S. Cynk and C. Meyer, Modular Calabi-Yau threefolds of level eight, preprint, math.AG/0504070. 2005.
- [CM05c] S. Cynk and C. Meyer, Modularity of some non-rigid double octic Calabi-Yau threefolds, preprint, math.AG/0505670. 2005.
- [Del71] P. Deligne, Formes modulaires et représentations l -adiques, In *Sem. Bourbaki* **355** (1968/69), Lect. Notes Math **179** (1971).
- [Del74] P. Deligne, La conjecture de Weil. I, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307.
- [Del80] P. Deligne, La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.* **52**, (1980), 137–252.
- [Deu53] M. Deuring, Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins. I, II, III, IV *Nachr. Akad. Wiss. Göttingen* 1953, 1955, 1956, 1957.
- [Di04a] L. Dieulefait, Computing the level of a modular rigid Calabi-Yau threefold, *Experiment. Math.* **13** (2004), 165–169.
- [Di04b] L. Dieulefait, From potential modularity to modularity for integral Galois representations and rigid Calabi-Yau threefolds, preprint, math.NT/0409102, to appear in *AMS/IP Studies in Advanced Mathematics “Mirror Symmetry V”* as Appendix to the paper On the modularity of Calabi-Yau threefolds containing elliptic ruled surfaces by K. Hulek and H. Verrill.
- [Di05] L. Dieulefait, Improvements on Dieulefait-Manoharmayum and applications, preprint, math.NT/050816. 2005.
- [DM03] L. Dieulefait and J. Manoharmayum. Modularity of rigid Calabi-Yau threefolds over \mathbf{Q} , In *Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001)*, Fields Inst. Commun. **38**, Amer. Math. Soc., Providence, RI, 2003, 159–166.
- [Dwo60] B. Dwork, On the rationality of the zeta function of an algebraic variety. *Amer. J. Math.* **82** (1960), 631–648.
- [Esn05] H. Esnault, Hodge type over the complex numbers and congruences for the number of rational points over finite fields, *Oberwolfach Report 7/2005*, 414–415.
- [Fal83] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Inv. Math.* **73** (1983), 349–366.
- [FO94a] Le corps des périodes p -adiques. In *Périodes p -adiques (Bures-sur-Yvette, 1988)*. Astérisque **223**, (1994), 59–113.
- [FO94b] Représentations p -adiques semi-stables. In *Périodes p -adiques (Bures-sur-Yvette, 1988)*. Astérisque **223** (1994), 113–184.
- [FM95] J.M. Fontaine and M. Mazur, Geometric Galois representations, In *Elliptic curves, Modular Forms and Fermat’s Last Theorem* (Hong Kong, 1993), J. Coates, S.T. Yau (eds.), International Press (1995), 41–78.
- [FK88] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjectures*, Springer Verlag, 1988.
- [FW05] L. Fu and D. Wan, Mirror congruence for rational points on Calabi-Yau varieties, preprint math.AG/0503703, 2005.
- [vGN95] B. van Geemen and N. O. Nygaard, On the geometry and arithmetic of some Siegel modular threefolds, *J. Number Theory* **53** (1995), 45–87.

- [Gri69] P. Griffiths, On the periods of certain rational integrals. I, II. *Ann. of Math. (2)* **90** (1969), 460-495; *ibid.* (2) **90** 1969, 496-541.
- [HSGS01] K. Hulek, J. Spandaw, B. van Geemen and D. van Straten, The modularity of the Barth-Nieto quintic and its relatives, *Adv. Geom.* **1** (2001), 263-289.
- [HV03] K. Hulek and H. Verrill, On modularity of rigid and nonrigid Calabi-Yau varieties associated to the root lattice A_4 , *Nagoya Math. Journal* **179** (2005) 103-146.
- [HV05a] K. Hulek and H. Verrill, On the modularity of Calabi-Yau threefolds containing elliptic ruled surfaces, preprint, math.AG/0502158, (to appear in *AMS/IP Studies in Advanced Mathematics "Mirror Symmetry V"*.) 2005.
- [HV05b] K. Hulek and H. Verrill, On the motive of Kummer varieties associated to $\Gamma_1(7)$ - Supplement to the paper: The modularity of certain non-rigid Calabi-Yau threefolds (by R. Livné and N. Yui), accepted by *Kyoto J. of Math.*, math.AG/0506388, 2005.
- [Jon] J. Jones, *Tables of number fields with prescribed ramification*, <http://math.la.asu.edu/~jj/numberfields>.
- [Kad04] S. Kadir, The Arithmetic of Calabi-Yau manifolds and mirror symmetry, hep-th/0409202, 2004.
- [Kat68] N. Katz, On the differential equation satisfied by period matrices, *Inst. Hautes Études Sci. Publ. Math.*, **35** (1968), 71-106.
- [Ked01] K. Kedlaya, Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology. *J. Ramanujan Math. Soc.* **16** (2001), 323-338.
- [Kis05] M. Kisin, Modularity of some geometric Galois representations, <http://www.math.uchicago.edu/~kisin/preprints.html>.
- [Kna92] A. W. Knap, *Elliptic curves*, Mathematical Notes **40**, Princeton Univ. Press, 1992.
- [Liv87] R. Livné, Cubic exponential sums and Galois representations, In *Current trends in arithmetical algebraic geometry (Arcata, Calif. 1985)*, Contemp. Math. **67**, K. Ribet (ed.), Amer. Math. Soc., Providence R.I. (1987), 247-261.
- [Liv95] R. Livné, Motivic orthogonal two-dimensional representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *Israel J. Math.* **92** (1995), 149-156.
- [LY03] R. Livné and N. Yui, The modularity of certain non-rigid Calabi-Yau threefolds, accepted by *Kyoto J. of Math.* math.AG/0304497. 2003.
- [Mar96] Y. Martin, Multiplicative η -quotients, *Trans. Amer. Math. Soc.* **348** (1996), 4825-4856.
- [Mey05] C. Meyer, *Modular Calabi-Yau threefolds*, Fields Institute Monographs, American Mathematical Society, Providence, 2005.
- [Ogg67] A.P. Ogg, Elliptic curves and wild ramification. *Amer. J. Math.* **89** (1967), 1-21.
- [vdP86] M. van der Put, The cohomology of Monsky and Washnitzer, *Mémoires de la S.M.F. (2)*. **23** (1986), 33-59.
- [Rib77] K. Ribet, Galois representations attached to eigenforms with Nebentypus, In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, 17-51. Lecture Notes in Math., **601**, Springer, Berlin, 1977.
- [Tay04] R. Taylor, Galois representations. *Ann. Fac. Sci. Toulouse Math. (6)* **13** (2004), 73-119.
- [SY01] M.-H. Saito and N. Yui, The modularity conjecture for rigid Calabi-Yau threefolds over \mathbb{Q} , *J. Math. Kyoto Univ.* **41** (2001), 403-419.
- [Scho86] C. Schoen, On the geometry of a special determinantal hypersurface associated to the Mumford-Horrocks bundle, *J. Reine Angew. Math.* **364** (1986), 85-111.
- [Scho88] C. Schoen, On fiber products of rational elliptic surfaces with section, *Math. Z.* **197** (1988), 177-199.
- [Schu04a] M. Schütt, New examples of modular rigid Calabi-Yau threefolds. *Collect. Math.*, **55** (2004), 219-228.
- [Schu04b] M. Schütt, On the modularity of three Calabi-Yau threefolds with bad reduction at 11, Accepted by *Canad. Math. Bull.*, math.AG/0405450, 2004.
- [Schu05] M. Schütt, Hecke eigenforms with rational coefficients and complex multiplication, preprint math.NT/0511228, 2005.
- [ST05] M. Schütt and J. Top, Arithmetic of the $[19, 1, 1, 1, 1, 1]$ fibration, preprint, math.AG/0510063, 2005.
- [Ser85] J.-P. Serre, Résumé des cours de 1984-1985, *Annuaire du Collège de France (1985)*, 85-90. (*Oeuvres IV*, 27-32.)

- [Ser87] J.-P. Serre, Sur les représentations modulaires de degré 2 de $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *Duke Math. J.* **54** (1987), 179–230.
- [Sha96] I. R. Shafarevich, *On the arithmetic of singular K3-surfaces*, Algebra and analysis (Kazan, 1994), 103–108, de Gruyter, Berlin, 1996.
- [Shi72] T. Shioda, On elliptic modular surfaces. *J. Math. Soc. Japan* **24** (1972), 20–59.
- [SI77] T. Shioda and H. Inose, On singular K3 surfaces, In *Complex analysis and algebraic geometry*, Iwanami Shoten, Tokyo, 1977, 119–136.
- [SM74] T. Shioda, N. Mitani, Singular abelian surfaces and binary quadratic forms. In *Classification of algebraic varieties and compact complex manifolds*, 259–287. Lecture Notes in Math. **412**, Springer, Berlin, 1974.
- [Sil94] J. H. Silverman *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics **151**, Springer Verlag, 1994.
- [St] W. A. Stein, *The modular forms database*, <http://modular.ucsd.edu>.
- [Ver00] H. A. Verrill, The L -series of Certain Rigid Calabi-Yau Threefolds, *J. Number Theory* **81** (2000), 310–334.
- [Voi93] C. Voisin, Miroirs et involutions sur les surfaces K3, In: *Journées de Géométrie Algébrique d'Orsay (Orsay, 1992)*. Astérisque **218** (1993), 273–323.
- [Wan04] D. Wan, Mirror symmetry for zeta functions, preprint, math.AG/0411464, 2004
- [Weil49] A. Weil, Number of solutions of equations over finite fields, *Bull. Amer. Math. Soc.* **55** (1949), 497–508.
- [Wein73] P. J. Weinberger, Exponents of the class groups of complex quadratic fields, *Acta Arith.* **22** (1973), 117–124.
- [WvG90] J. Werner and B. van Geemen, New examples of threefolds with $c_1 = 0$, *Math. Z.* **203** (1990), 211–225.
- [Wil95] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math. (2)* **141** (1995) 443–551.
- [Yi05] Y.-C. Yi, On the Hilbert modularity of a certain quintic threefold, Preprint 2005.
- [Yui03] N. Yui, Update on the modularity of Calabi-Yau varieties, With an appendix by Helena Verrill. In *Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001)*, Fields Inst. Commun., **38**, 307–336, Amer. Math. Soc., Providence, RI, 2003.

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